more than one !

Ch. 11.1 Function of Several Variables

Remember: $y = f(x)$ → one variable function.

dependent variable y. | independent variable x.

$z = f(x,y)$ → two variables function

dependent variable z. | independent variables x and y.

$w = f(x,y,z)$ → three variables function

dependent variable w. | independent variables x,y,z.

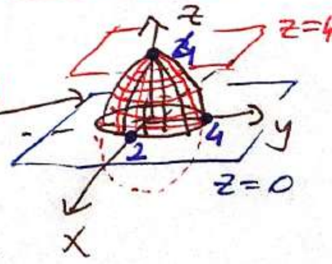
A function of $f(x,y)$ is a rule that associates for each point (x,y) in the domain D a unique value $f(x,y)$ in the range R.

Ex. Find the domain and the range of

$$z = f(x,y) = \sqrt{16-4x^2-y^2}.$$

≥ 0 (in order to square root to be defined)

upper ellipsoid



we must have

$16-4x^2-y^2 \geq 0$

$\Rightarrow 4x^2+y^2 \leq 16$

$\Rightarrow \dfrac{x^2}{\frac{1}{4}} + y^2 \leq 16$

$\Rightarrow \dfrac{x^2}{4} + \dfrac{y^2}{16} \leq 1$

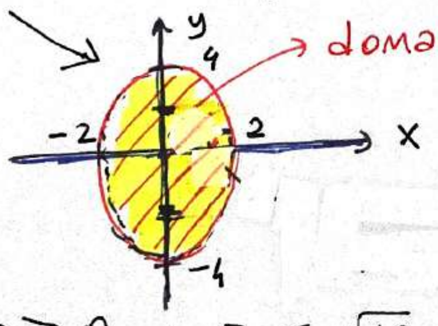
or equivalently $\Rightarrow \boxed{\dfrac{x^2}{2^2} + \dfrac{y^2}{4^2} \leq 1}$

{ boundary and inside the upper ellipsoid }

Then,

$D = \{(x,y) \mid \frac{x^2}{2^2} + \frac{y^2}{4^2} \leq 1,\ x,y \in \mathbb{R}\}$

such that



$R: z \geq 0,\ z \leq \sqrt{16} = 4.$

$z = f(0,0) = \sqrt{16-4\cdot 0^2-0^2} = 4$ → Max. value.

$R: \underset{\text{min}}{0} \leq z \leq \underset{\text{max}}{4}$ $\{z \in [0,4]\}$

Ex. Let $f(x,y) = \sin(3x+4y)$. Find $f(1,2)$.
What is the range of $f(x,y)$?

$$f(1,2) = \sin(3\cdot(1)+4(2)) = \sin(3+8) = \sin(11) \hat{=} -0.999.$$

Since $f(x,y) = \sin(3x+4y)$ is a sine function, the range of the sine function remains between -1 and 1, inclusive. Then $R = [-1,1] = \{(x,y) \mid -1 \leq x \leq 1$ and $-1 \leq y \leq 1, (x,y) \in \mathbb{R}\}$

Ex. Let $f(x,y) = 2 + \sqrt{9-y^2}$. Evaluate $f(4,2)$ and find the range of $f(x,y)$.

$$\left.\begin{array}{l} x=4 \\ y=2 \end{array}\right\} \Rightarrow f(4,2) = 2+\sqrt{9-(2)^2} = 2+\sqrt{5}$$

To find the range of $f(x,y)$, we consider the domain of y since the expression inside the square root must be non-negative. Then,

$$\Rightarrow \sqrt{9-y^2} \geq 0 \quad \Rightarrow 9-y^2 \geq 0 \quad \Rightarrow y^2 \leq 9 \Rightarrow -3 \leq y \leq 3$$

The minimum value of $\sqrt{9-y^2}$ occurs when $y = \mp 3$. That is,

$$\sqrt{9-(3)^2} = \sqrt{0} = 0.$$

Hence, at $y = \mp 3$; $f(x, \mp 3) = 2+0 = 2$

The maximum value of $\sqrt{9-y^2}$ occurs when $y=0$. That is,

$$\sqrt{9-(0)^2} = \sqrt{9} = 3.$$

Hence, at $y=0$; $f(x,0) = 2+3 = 5$

Thus, $R = [2,5] = \{(x,y) \mid 2 \leq y \leq 5, (x,y) \in \mathbb{R}\}$

Ex. Let $f(x,y,z) = \sqrt{x^3y^2z^4 - x - y - z}$.

Evaluate $f(2,3,4)$, $f(-1,-4,2)$.

- $f(2,3,4) = \sqrt{2^3 . 3^2 . 4^4 - 2 - 3 - 4} = \sqrt{72.256 - 9} = \sqrt{18423}$

$(x=2, y=3, z=4)$ — defined in $\mathbb{R}$.

- $f(-1,-4,2) = \sqrt{(-1)^3 . (-4)^2 . (2)^4 - (-1) - (-4) - 2}$

$(x=-1, y=-4, z=2)$

$= \sqrt{(-1)\,16.16 + 1 + 4 - 2}$

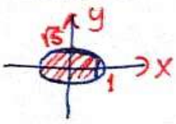
$= \sqrt{-256 + 3}$

$= \sqrt{-253}$

So, $f(-1,-4,2)$ is not defined in $\mathbb{R}$. But defined in $\mathbb{C}$.

$\boxed{\text{real numbers}}$ $\boxed{\text{complex numbers.}}$

Ex. $f(x,y) = \sqrt{5 - 5x^2 - y^2}$ $D = ?$, $R = ?$

D: 

$5 - 5x^2 - y^2 \geq 0 \Rightarrow 5 \geq 5x^2 + y^2 \Rightarrow x^2 + \frac{y^2}{5} \leq 1$

gives an ellipse centered $C(0,0)$ with semi major-minor axes $1, \sqrt{5}$, resp., along x and y-axes, resp.

The maximum value of $f(x,y)$ occurs at $C(0,0)$ of the ellipse: $f(0,0) = \sqrt{5 - 5.(0)^2 - (0)^2} = \sqrt{5}$ (max. value)

The minimum value of $f(x,y)$ occurs at the boundary where $5 - 5x^2 - y^2 = 0 \Rightarrow f(x,y) = 0$

Therefore, $R = [0, \sqrt{5}]$

$x \in \mathbb{R}, y \in \mathbb{R}$

$D = \left\{ (x,y) \mid x^2 + \frac{y^2}{5} \leq 1,\ (x,y) \in \mathbb{R}^2 \right\}$

Ex. $f(x,y) = \sqrt{2x^2+y^2}$ $D = ?$ $R = ?$

$2x^2+y^2$ is always positive. That is, $2x^2+y^2 \geq 0$.
Non-negative. So, $D = \mathbb{R}^2$ ($\forall (x,y)$ pairs)

The minimum value occurs at $(0,0)$:

$$f(0,0) = \sqrt{2.(0)^2+(0)^2} = 0.$$

As (x,y) increase, $f(x,y)$ can become arbitrary large.
Therefore, $R = [0, \infty)$.

Ex. Find domain D and range R for the following:

$f(x,y) = 3 + \sqrt{4x^2+3y^2}$,
$g(x,y) = 4 + \sqrt{y}$,
$h(x,y) = 2 + \sqrt{3x^2+y^2}$.

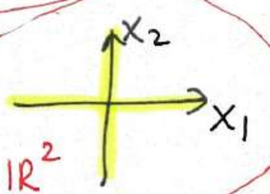
Ex. Find the following for $f(x,y) = \sin(x^2+y^2) + e^{2xy}$

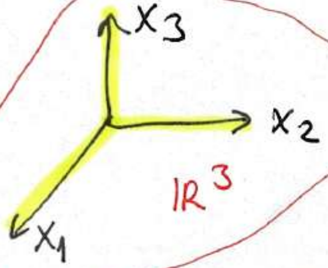
$f(\pi, 2\pi)$,
$f(0,0)$,
$f(1,1)$,
$f(-2,0)$,
$f(e,e)$.

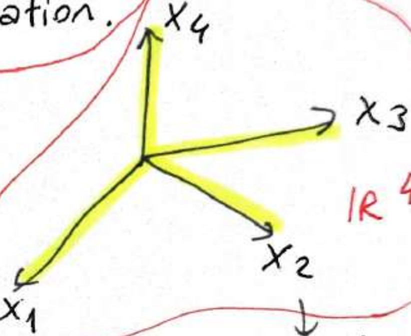
$$ax+by+c=0$$
a line equation
(or $ax_1+bx_2+c=0$)

$$AX+BY+Cz+D=0$$
a plane equation
(or ($AX_1+BX_2+CX_3+D=0$))

$$Ax+By+Cz+Dw+E=0$$
a hyperplane equation.
(or ($AX_1+BX_2+CX_3+DX_4+E=0$))







just imagine $\mathbb{R}^4$!

$$x^2+y^2=r^2$$
(or ($x_1^2+x_2^2=r^2$))
a circle equation

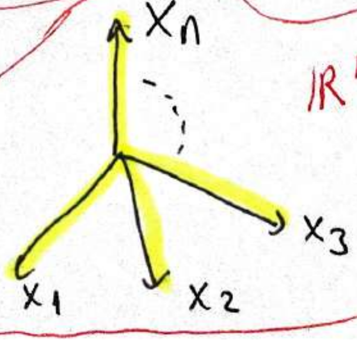
$$x^2+y^2+z^2=r^2$$
(or $x_1^2+x_2^2+x_3^2=r^2$)
a sphere equation

$$x_1^2+x_2^2+x_3^2+x_4^2=r^2$$
a hyperspace in 4D

We can not draw it! in 4D.

$\sum_{i=1}^{n}(x_i)^2=r^2$ hypersphere in nD

$$a_1x_1+a_2x_2+\ldots+a_nx_n+a_{n+1}=0$$
hyperplane in nD.
(or $\sum_{i=1}^{n} a_ix_i+a_{n+1}=0$)



Assume $f(x,y)$ a function, $g(x,y)$ a function defined on D_1 and D_2, respectively. Then,

sum $(f+g)(x,y) = f(x,y) + g(x,y)$

difference $(f-g)(x,y) = f(x,y) - g(x,y)$

product $(f\cdot g)(x,y) = f(x,y)\cdot g(x,y)$

$\Big\}$ $\forall (x,y)$ in $\{D_1 \cap D_2\}$ → for all

quotient $\left(\frac{f}{g}\right)(x,y) = \dfrac{f(x,y)}{g(x,y)}$ $\Big\}$ $\forall (x,y)$ in $\{D_1 \cap D_2\} \setminus \{(x,y) \mid g(x,y)=0\}$

A function of two variables is a rule, denoted by f, that assigns a uniqu value $f(x,y)$ to each ordered pair (x,y) in a set D. The set D is called the domain of the function, while the resulting values $f(x,y)$ of the function form the range of the function

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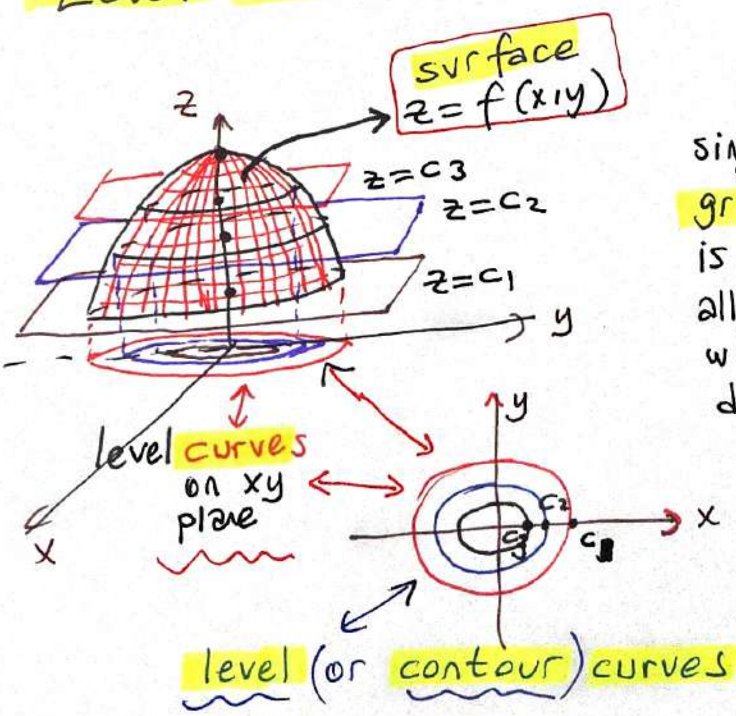
Level Curves and Surfaces



Similar to the single-variable case, the graph of a function $f(x,y)$ is defined as the set of all ordered triples (x,y,z), where (x,y) lies within the domain of f and $z=f(x,y)$.

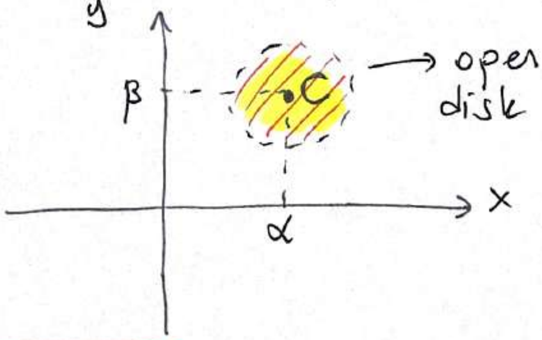
The graph forms a surface in $\mathbb{R}^3$, with the domain D projected onto the xy plane.

Ch 11.2 Limits and Continuity

Open and Closed Sets in 2D and 3D.

In $\mathbb{R}^2$, $\boxed{\text{Open Disk}}$: It centered at $C(\alpha,\beta)$ in $\mathbb{R}^2$, and set of all points $Q(x,y)$ such that

$$\sqrt{(x-\alpha)^2+(y-\beta)^2} < r, \quad r>0.$$



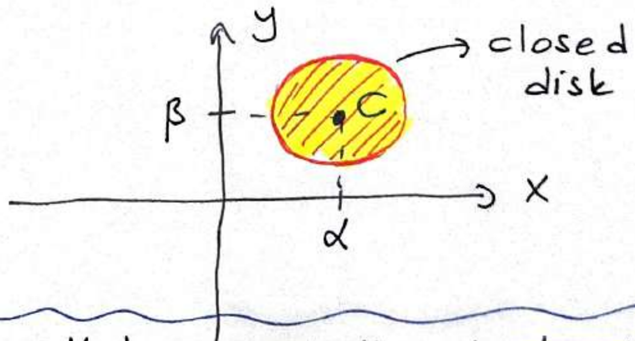
$$(x-\alpha)^2+(y-\beta)^2 < r^2.$$

In $\mathbb{R}^2$, $\boxed{\text{Closed Disk}}$: When the boundary of the disk is included (i.e., if $(x-\alpha)^2+(y-\beta)^2 \le r^2$), the disk is referred to as closed.

center $C(\alpha,\beta)$

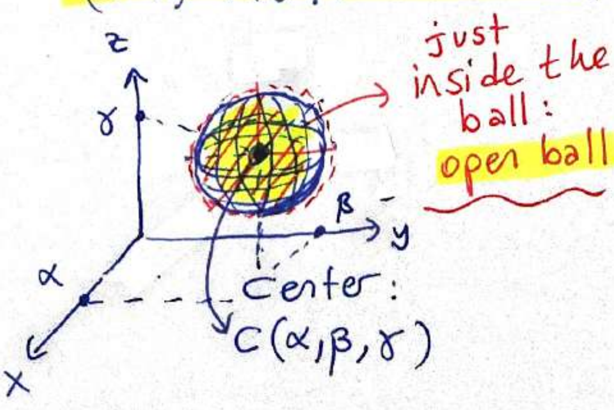
$$(x-\alpha)^2+(y-\beta)^2 \le r^2.$$

$r>0$

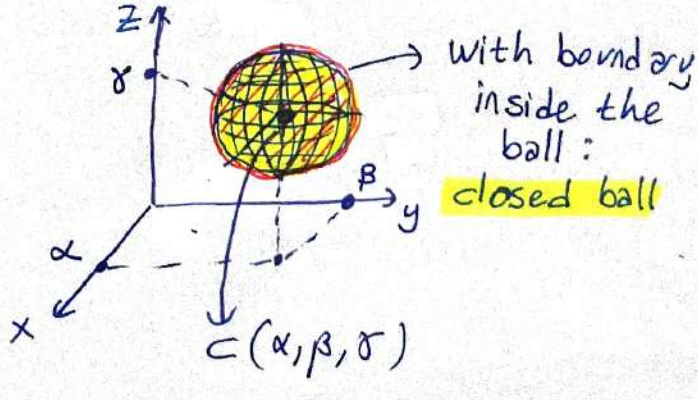


Similarly, in $\mathbb{R}^3$: they are called open ball and closed ball.

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2 < r^2$$



$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2 \le r^2$$



Limit of a Function of Two Variables

$$\left(\boxed{\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L}\right) \Longleftrightarrow \left(\boxed{\begin{array}{l}\forall \varepsilon > 0\ ;\ \exists\, \delta > 0 \text{ such that} \\ 0 < (x-x_0)^2 + (y-y_0)^2 < \delta, \text{ it follows that} \\ |f(x,y) - L| < \varepsilon .\end{array}}\right)$$

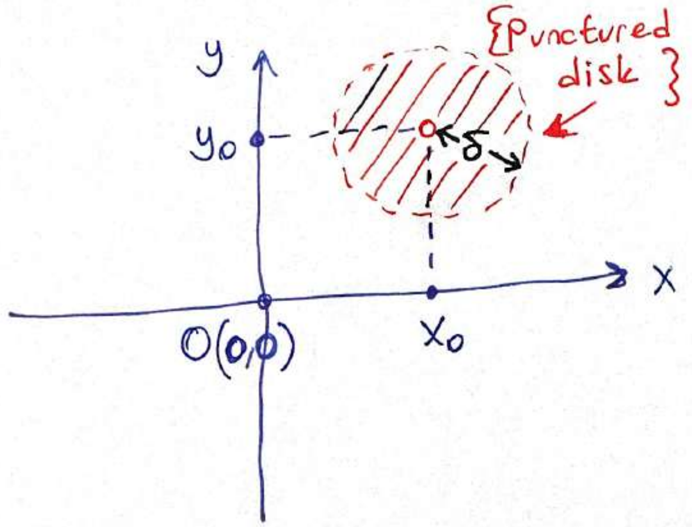
{ definition of limit mathematically. }

The limit expression $\lim f(x,y)$ as (x,y) approaches (x_0,y_0) indicates that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever (x,y) is a point within the domain D of f and satisfies the condition

$$0 < (x-x_0)^2 + (y-y_0)^2 < \delta ,$$

it follows that

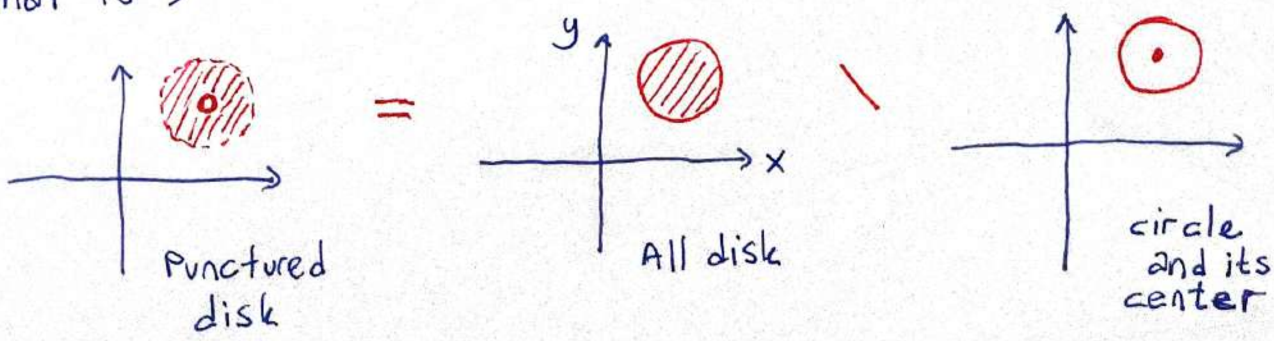
$$|f(x,y) - L| < \varepsilon .$$



{Punctured disk} = {All disk with boundary and inside points} \ {Boundary and (x_0,y_0)}

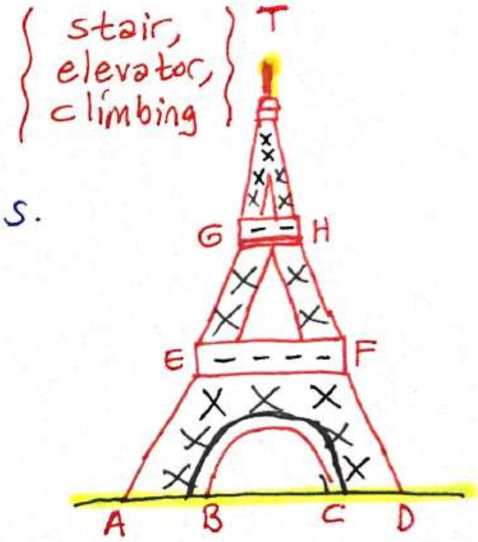
(\ : difference)

That is ;



When examining limit $f(x)$, we consider the approach of x towards c from two directions (left-hand and right-hand limits). However, for a function of two variables, when we write $(x,y) \longrightarrow (x_0, y_0)$, it means that the point (x,y) is allowed to approach (x_0,y_0) along any curve within the domain of f that passes through (x_0,y_0).

For example, no matter which path you take to reach the top of the Eiffel Tower, once you arrive, you'll see the same view of Paris.



If $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

is not the same for all paths within the domain of f, then the limit does not exist (DNE)

Ex. Find the limit: $\displaystyle\lim_{(x,y)\to(0,0)} \underbrace{\frac{\sin[5(x^2+y^2)]}{5(x^2+y^2)}}_{f(x,y)}$.

$\left\{ \text{Hint: } \displaystyle\lim_{t\to 0} \frac{\sin t}{t} = 1. \right\}$

In polar coordinates $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} : x^2+y^2 = r^2$

$f(r,\theta) = \dfrac{\sin(5r^2)}{5r^2}$. $(x,y) \longrightarrow (0,0) \Rightarrow r \longrightarrow 0$.

$t = 5r^2 \Rightarrow r \longrightarrow 0$ then $t \longrightarrow 0$: $\displaystyle\lim_{r\to 0} \frac{\sin(5r^2)}{5r^2} = \lim_{t\to 0} \frac{\sin t}{t} = 1$.

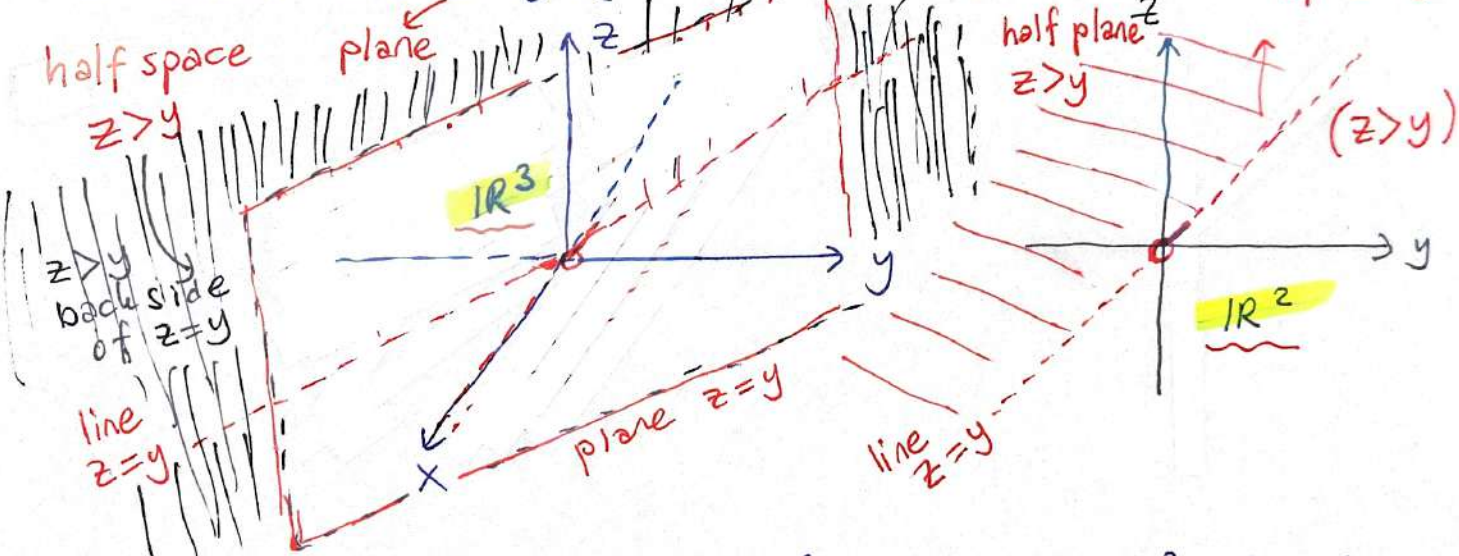
Ex. Find the domain of

$$f(x,y,z) = \ln(z-y) + xy\sin z .$$

$z - y > 0 \Rightarrow f(x,y,z)$ is a defined function.

$$\Rightarrow D = \{ (x,y,z) \in \mathbb{R}^3 \mid z > y \}$$

All above the points of $z=y$ (which is a plane). upper part of $z=y$



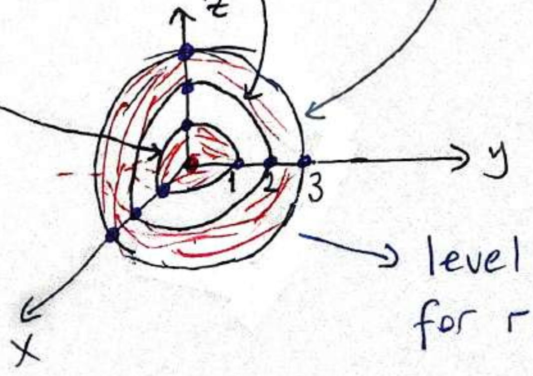
Ex. Find the level surfaces of $f(x,y,z) = x^2+y^2+z^2$

The level surfaces are $x^2+y^2+z^2 = c$ (surfaces)

$(c \geq 0)$ The spheres have radius $\sqrt{c}$.

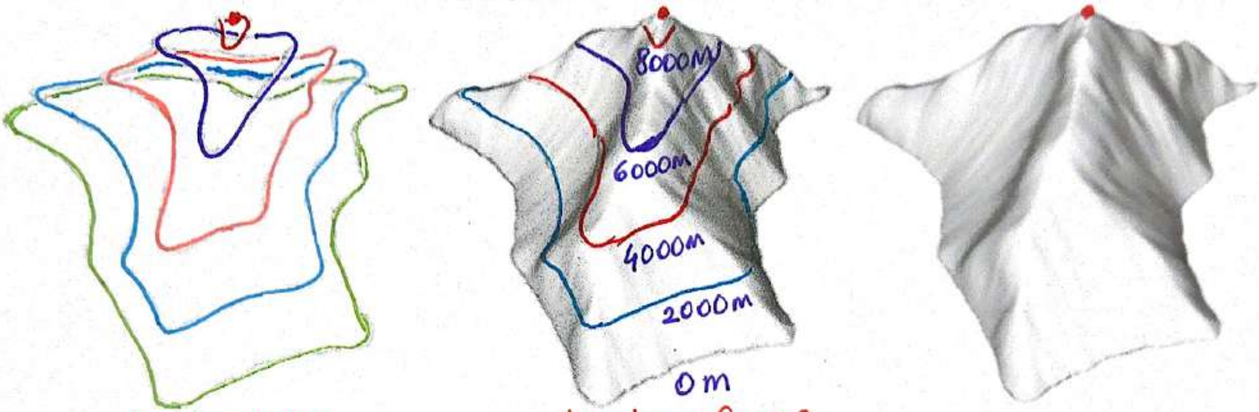
$c=1$, $c=2$, $c=3$

$x^2+y^2+z^2=1$ $\quad$ $x^2+y^2+z^2=4$ $\quad$ $x^2+y^2+z^2=9$



level surfaces (spheres) for $r=1,2,3$ {in first quadrant}

(summit Everest) highest point in the world
8848 m

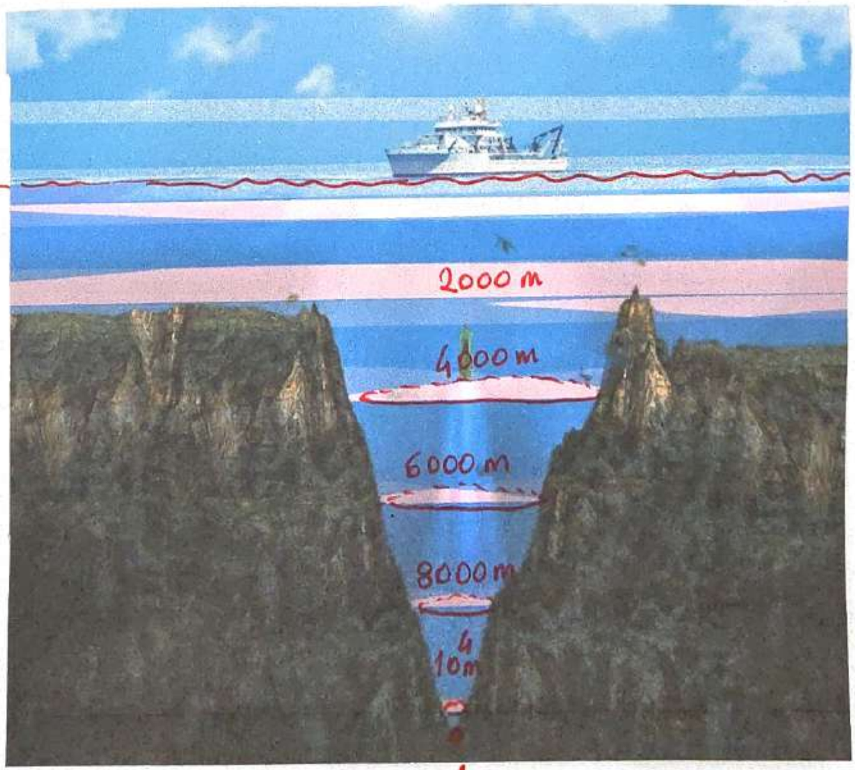


level curves level surfaces

Level curves and surfaces of Mt. Everest.

(0 m means sea level)

0 m sea level



deepest point
in the world

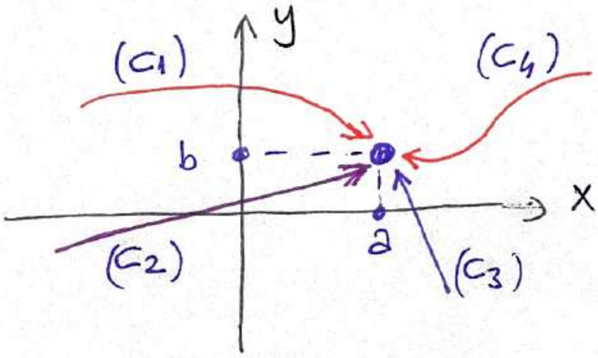
Mariana Trench

(10 935 m below)

If, along path (C_1), as $(x,y) \to (a,b)$ as $f(x,y) \to L_1$ and along path (C_2), as $(x,y) \to (a,b)$ as $f(x,y) \to L_2$ then

$$\boxed{\lim_{(x,y)\to(a,b)} f(x,y)}$$

does not exist (DNE). Here, $L_1 \neq L_2$.



$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

For the functions of one variable we can approximate x to a from only two directions, from the right or the left.

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L \overset{\text{iff}}{\Longleftrightarrow} \lim_{x\to a} f(x) = L.$$

The situation is not so simple for the functions of two variables, because (x,y) can be approached to (a,b) from infinitely many directions ((x,y) provided that it remains in the domain of f). (see figure above)

Ex. Let $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$. Find the limit :

$$\lim_{(x,y)\to(0,0)} f(x,y).$$

First, let us approach $(0,0)$ along the x-axis.
In this case, $y=0$ for all $x \neq 0$,

$$f(x,0) = \frac{x^2-(0)^2}{x^2+(0)^2} = \frac{x^2}{x^2} = 1.$$

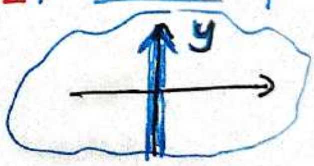
$\left\{ \begin{array}{c} C_1 \text{ path:} \\ \text{x-axis} \end{array} \right\}$

Then, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 1 = L_1$

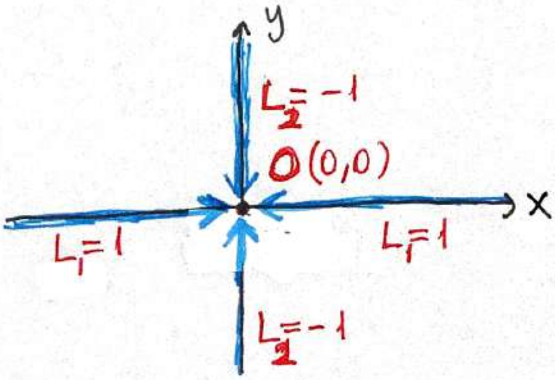
Now, let's approximate along the y-axis by taking $x=0$. In this case, for all $y \neq 0$.

$$f(0,y) = \frac{(0)^2 - y^2}{(0)^2+y^2} = \frac{-y^2}{y^2} = -1 = L_2$$

$\left\{ \begin{array}{c} C_2 \text{ path} \\ \text{y-axis} \end{array} \right\}$



Then, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ DNE. $(L_1 \neq L_2)$



Ex. Let $f(x,y) = \dfrac{xy}{x^2+y^2}$. Find the limit:

$$\lim_{(x,y)\to(0,0)} f(x,y).$$

If $y=0$, then $f(x,0) = \dfrac{x.(0)}{x^2+(0)^2} = \dfrac{0}{x^2} = 0$.

Then, along the x-axis

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = L_1$$

$\left\{ \boxed{c_1 \text{ path: } x\text{-axis}} \right\}$.

If $x=0$, then $f(0,y) = \dfrac{(0).y}{(0)^2+y^2} = \dfrac{0}{y^2} = 0$.

Then, along the y-axis

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = L_2$$

$\left\{ \boxed{c_2 \text{ path: } y \text{ axis}} \right\}$

Although we obtained the same limit along the axes, this result does not indicate that the limit is 0.

Let's approach $(0,0)$ along another line, for example the line $y=x$.

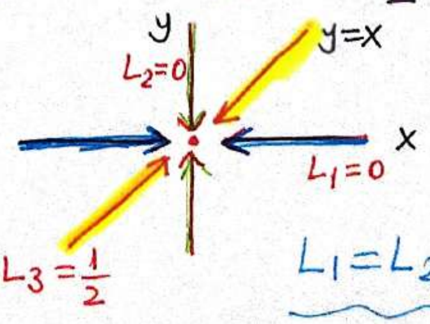
$\left\{ \boxed{c_3 \text{ path: line } y=x} \right\}$

For all $x \neq 0$

$$f(x,x) = \frac{x.x}{x^2+x^2} = \frac{x^2}{2x^2} = \frac{1}{2} = L_3$$

Therefore, along the line $x=y$, $f(x,y) \to \frac{1}{2}$ as $(x,y)\to(0,0)$.

$\left\{ \text{Since we obtain different limits along different paths, limit DNE.} \right\}$



$L_1 = L_2 \neq L_3$

Ex. Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{5x^2}{2x^2+4y^2}$.

- Path along $y=0$:

$$\lim_{(x,y)\to(0,0)} \frac{5x^2}{2x^2+4.(0)^2} = \lim_{(x,y)\to(0,0)} \frac{5x^2}{2x^2} = \frac{5}{2}.$$

- Path along $x=0$:

$$\lim_{(x,y)\to(0,0)} \frac{5.(0)^2}{2.(0)^2+4y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{4y^2} = 0.$$

$\neq$

- Path along $y=mx$ (where m is a constant):

$$\lim_{(x,y)\to(0,0)} \frac{5x^2}{2x^2+4(mx)^2} = \lim_{x\to 0} \frac{5x^2}{(2+4m^2)x^2} = \lim_{x\to 0} \frac{5}{2+4m^2}$$

If $m=0$, $\lim\limits_{x\to 0} \dfrac{5}{2+4.(0)^2} = \dfrac{5}{2}$,

If $m=1$, $\lim\limits_{x\to 0} \dfrac{5}{2+4.(1)^2} = \dfrac{5}{6}$.

$\neq$

Since the limits along different paths yields different results, the limit does not exist overall.

Essential Formulas and Principles of Limits with Two Variables

Assume $\lim\limits_{(x,y)\to(x_0,y_0)} h(x,y) = L$ and $\lim\limits_{(x,y)\to(x_0,y_0)} k(x,y) = M$.

For any constant a, the following principles hold:

- $\lim\limits_{(x,y)\to(x_0,y_0)} [a.h(x,y)] = a.\lim\limits_{(x,y)\to(x_0,y_0)} h(x,y) = a.L$, (constant multiple / scalar multiple)
- $\lim\limits_{(x,y)\to(x_0,y_0)} [h(x,y)+k(x,y)] = L+M$, (Addition (Sum))
- $\lim\limits_{(x,y)\to(x_0,y_0)} [h(x,y).k(x,y)] = L.M$, (multiplication (Product))
- $\lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{h(x,y)}{k(x,y)} = \dfrac{L}{M}$, assuming $M \neq 0$. ((Quotient) Divison)

Continuity:

A function of one variable $f(x)$ is continuous at $x=c$ if:

a • $f(c)$ is defined,

b • $\lim\limits_{x\to c} f(x)$ exists,

c • $\lim\limits_{x\to c} f(x) = f(c)$.

$\begin{pmatrix} f(x) \text{ is continuous} \\ \text{at } x=c \end{pmatrix} \overset{\text{iff}}{\Longleftrightarrow} \begin{pmatrix} a,b,c \\ \text{hold.} \end{pmatrix}$

Similarly, a function of two variables $f(x,y)$ is continuous at (x_0,y_0) if:

α • $f(x_0,y_0)$ is defined,

β • $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists,

γ • $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

$\begin{pmatrix} f(x,y) \text{ is continuous} \\ \text{at } (x_0,y_0) \end{pmatrix} \overset{\text{iff}}{\Longleftrightarrow} \begin{pmatrix} \alpha,\beta,\gamma \\ \text{hold} \end{pmatrix}$

Ex. Let $f(x,y) = x^2+y^2$.

The function is continuous at every point, as it satisfies $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$. For istance,

at the point $(0,0)$:

$$f(0,0) = 0^2+0^2 = 0 \,, \quad \lim_{(x,y)\to(0,0)} (x^2+y^2) = 0$$

Hence, the function is continuous at every point (x_0,y_0)

Ex. Let $h(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ (for $(x,y) \neq (0,0)$).

$h(x,y)$ is not defined at the point $(0,0)$ and has different limits approaching this point from different paths, making it discontinuous. For example:

- along the line $y=0$: $\lim_{x\to 0} \dfrac{x^2-0^2}{x^2+0^2} = 1$,
- along the line $x=0$: $\lim_{y\to 0} \dfrac{0^2-y^2}{0^2+y^2} = -1$.

Therefore, the function is discontinuous at $(0,0)$.

Continuous can be defined for the functions $k(x,y,z)$ as in $f(x,y)$.

It also can be defined for more than three variables.

Ch. 11.3 Partial Derivatives

Understanding how a function of two variables changes with respect to one variable is often important.

For example in physics, consider the height h of an object as a function of horizontal distance x and time t, $h = f(x,t)$.

We might analyze how height changes with time while keeping distance constant, or how it changes with distance while keeping time constant. This differentiation with respect to one variable, while fixing the others, is called partial differential, and the resulting derivative is known as a partial derivative.

For one variable: $\boxed{f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}}$, $\left(y = f(x) \right)$

similarly, for two variables
partial derivatives are defined by

$$\frac{\partial f(x,y)}{\partial x} = f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}, \left(z = f(x,y) \right)$$

and

$$\frac{\partial f(x,y)}{\partial y} = f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}. \left(z = f(x,y) \right)$$

Notations:
for partial derivatives

$$\left\{ \begin{aligned} &\frac{\partial f(x,y)}{\partial x} = f_x(x,y) \\ &= D_x(f) = z_x = \frac{\partial z}{\partial x} \end{aligned} \right\} \left\{ \begin{aligned} &\frac{\partial f(x,y)}{\partial y} = f_y(x,y) \\ &= D_y(f) = z_y = \frac{\partial z}{\partial y} \end{aligned} \right\} z = f(x,y)$$

Ex. Let $f(x,y) = e^{2x+3y} + x^2y^3 - \sin(x-2y)$.

Find f_x and f_y.

- To find f_x, take y constant and then,

$$f_x = 2.e^{2x+3y} + 2xy^3 - 1.\cos(x-2y)$$

- Similarly, for f_y, take x constant:

$$f_y = 3.e^{2x+3y} + 3x^2y^2 - (-2)\cos(x-2y).$$

Hence;

$$f_x = 2e^{2x+3y} + 2xy^3 - \cos(x-2y),$$

$$f_y = 3e^{2x+3y} + 3x^2y^2 + 2\cos(x-2y).$$

Partial Derivative of an Implicit Function

Ex. Let $\sin(x^2+y^3) + x^3.z - y^2.z^2 = z$. Find z_x, z_y.

$$z_x = \frac{\partial z}{\partial x} = 2x\cos(x^2+y^3) + (3x^2z + x^3z_x) - (y^2.2z.z_x)$$

(consider y as a constant)

$$z_y = \frac{\partial z}{\partial y} = 3y^2\cos(x^2+y^3) + x^3z_y - (2yz^2 + y^2 2z.z_y)$$

(x as a constant)

Then,

$$2x\cos(x^2+y^3)+3x^2z = z_x\left(-x^3+2y^2z+1\right)$$

$$\Rightarrow z_x = \frac{2x\cos(x^2y^3)+3x^2z}{-x^3+2y^2z+1}$$

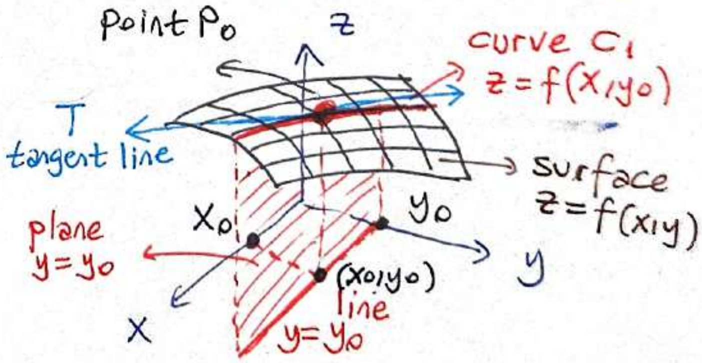
Similarly,

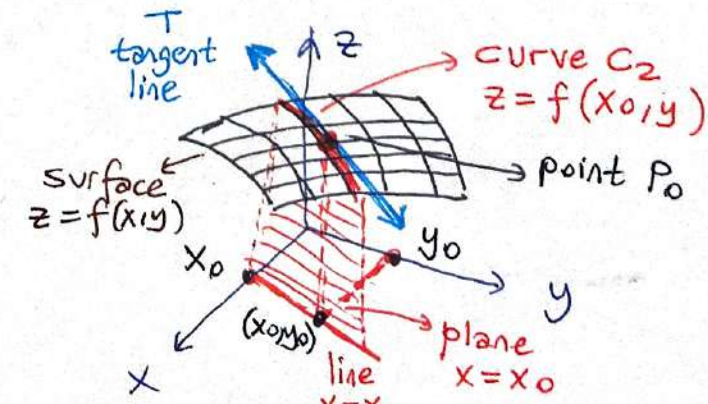
$$3y^2\cos(x^2+y^3)-2yz^2 = z_y\left(-x^3+2y^2z+1\right)$$

$$\Rightarrow z_y = \frac{3y^2\cos(x^2+y^3)-2yz^2}{-x^3+2y^2z+1}$$

Slope of a Tangent Line :

At a point $P_0(x_0,y_0,z_0)$ on the surface defined by $z=f(x,y)$, the line that is tangent to the surface and parallel to the xz-plane has a slope of $f_x(x_0,y_0)$. In constrast, the tangent line at P_0 that runs parallel to the yz-plane has a slope of $f_y(x_0,y_0)$.





Higher-Order Partial Derivatives

Let $z = f(x,y)$.

The second-order partial derivatives are determined by

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial x}\right) = (f_x)_x = f_{xx},$$

$$z_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial y}\right) = (f_y)_y = f_{yy},$$

$$z_{yx} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right) = (f_y)_x = f_{yx},$$

$$z_{xy} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) = (f_x)_y = f_{xy}.$$

Ex. For $z = f(x,y) = 2x^2 + 3x^3y^2 - 4y^4$, find $f_{xx}, f_{yy}, f_{xy}, f_{yx}$ at $(1,-1)$.

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(4x + 9x^2y^2\right) = 4 + 18xy^2$$

$$f_{xx}(1,-1) = (4(1) + 18(1)(-1)^2) = 4 + 18 = 22,$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(6x^3y - 16y^3\right) = 6x^3 - 48y^2$$

$$f_{yy}(1,-1) = 6.(1)^3 - 48(-1)^2 = 6 - 48 = -42,$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(4x + 9 \cdot x^2y^2\right) = 18x^2y$$

$$f_{xy}(1,-1) = 18.(1)^2(-1) = -18,$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(6x^3y - 16y^3\right) = 18x^2y$$

$$f_{yx}(1,-1) = 18.(1)^2(-1) = -18.$$

- If the mixed second-order derivatives f_{xy} and f_{yx} of the function $f(x,y)$ are continuous in a disk around (x_0,y_0), then $\boxed{f_{yx}(x_0,y_0)=f_{xy}(x_0,y_0)}$.

Ex. Let $f(x,y) = e^x \sin y + y^2 \cos x$.

$$f_x = e^x \sin y - y^2 \sin x,$$

$$f_y = e^x \cos y + 2y\cos x.$$

Then,

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(e^x \sin y - y^2 \sin x\right)$$

$$\Rightarrow f_{xy} = e^x \cos y - 2y\sin x.$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(e^x \cos y + 2y\cos x\right)$$

$$f_{yx} = e^x \cos y - 2y \sin x.$$

$\Rightarrow f_{xy} = f_{yx}$.

Ex. Let $f(x,y) = \ln(x^2+y^3) + \cos(2x - e^y)$.

Find f_{xy} and f_{yx}.

$$f_x = \frac{2x}{x^2+y^3} + 2.\left(-\sin(2x-e^y)\right)$$

$$(f_x)_y = \frac{\cancel{0.(x^2+y^3)} - 3y^2.2x}{(x^2+y^3)^2} + 2e^y\cos(2x-e^y)$$

$$f_y = \frac{3y^2}{x^2+y^3} + e^y \sin(2x - e^y)$$

$$(f_y)_x = \frac{\cancel{0.(x^2+y^3)} - 3y^2\,2x}{(x^2+y^3)^2} + 2e^y\cos(2x-e^y)$$

$f_{xy} = f_{yx}$

If the mixed third-order derivatives f_{xzy}, f_{yxz}, f_{zxy}, f_{xyz}, f_{yzx}, and f_{zyx} of the function $f(x,y,z)$ are continuous in a ball around (x_0,y_0,z_0), then

$$\boxed{f_{xyz}(x_0,y_0,z_0) = f_{yzx}(x_0,y_0,z_0) = f_{zyx}(x_0,y_0,z_0)}.$$
$$\boxed{= f_{xzy}(x_0,y_0,z_0) = f_{yxz}(x_0,y_0,z_0) = f_{zxy}(x_0,y_0,z_0)}$$

Ex. Let $f(x,y,z) = x^4y^2z^3 + x + y^5 + z^{10}$.

Compute mixed (all) third-order derivatives.

$$\left\{\begin{aligned} f_x &= 4x^3y^2z^3 + 1 \\ f_y &= 2x^4yz^3 + 5y^4 \\ f_z &= 3x^4y^2z^2 + 10z^9 \end{aligned}\right\} \quad \begin{aligned} (f_x)_y &= f_{xy} = 8x^3yz^3 \\ (f_{xy})_z &= f_{xyz} = 24x^3yz^2. \end{aligned}$$

$(f_x)_z = f_{xz} = 12x^3y^2z^2 \Rightarrow f_{xzy} = 24x^3yz^2.$

$(f_y)_x = f_{yx} = 8x^3yz^3 \Rightarrow f_{yxz} = 24x^3yz^2.$

$(f_y)_z = f_{yz} = 6x^4yz^2 \Rightarrow f_{yzx} = 24x^3yz^2.$

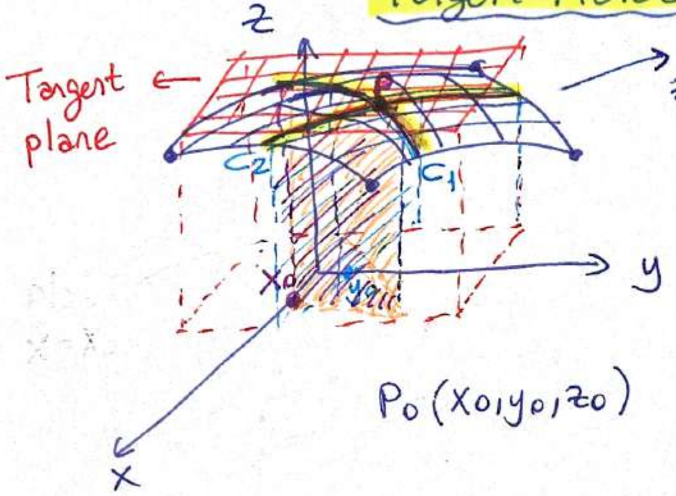
$(f_z)_x = f_{zx} = 12x^3y^2z^2 \Rightarrow f_{zxy} = 24x^3yz^2.$

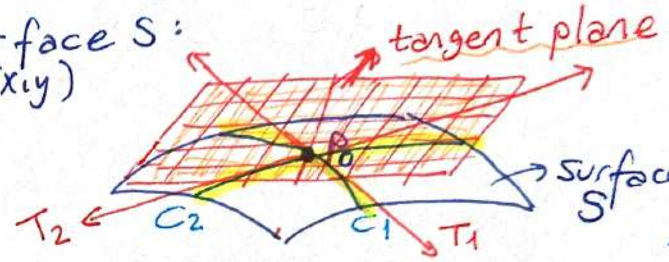
$(f_z)_y = f_{zy} = 12x^4yz^2 \Rightarrow f_{zyx} = 24x^3yz^2.$

Then, $f_{xyz} = f_{xzy} = f_{yxz} = f_{yzx} = f_{zxy} = f_{zyx}$.

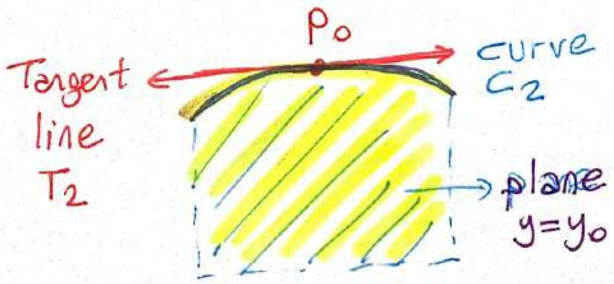
Ch. 11.4 Tangent Planes, Approximations, Differentiability

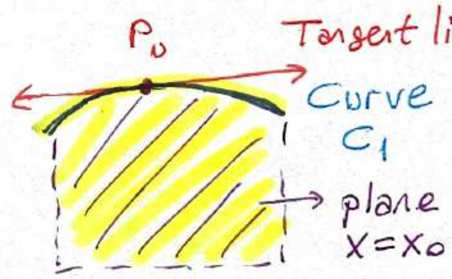
Tangent Planes:





Tangent plane intersects the surface only at point $P_0(x_0,y_0,z_0)$.





Side views of c_1 and c_2.

$z = f(x,y)$, $z_0 = f(x_0,y_0)$,

$P_0(x_0,y_0,z_0) = P_0(x_0, y_0, f(x_0,y_0))$ in 3D.

Suppose S is a surface defined by $z=f(x,y)$ with continuous partial derivatives f_x and f_y. At point $P_0(x_0,y_0,z_0)$, the curves formed by intersecting S with planes $x=x_0$ and $y=y_0$ have tangent lines T_1 and T_2, which determine a unique tangent plane at P_0. This plane contains the tangents to all smooth curves on S passing through P_0.

For a surface $z=f(x,y)$, the equation of the tangent plane at a point $P_0(x_0,y_0,z_0)$ is

$$\boxed{z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)}\,.$$

To convert this to standart form $Ax+By+Cz+D=0$:

$$A=f_x(x_0,y_0)\,,$$
$$B=f_y(x_0,y_0)\,,$$
$$C=-1$$
$$D=z_0-f_x(x_0,y_0)\,x_0-f_y(x_0,y_0)\,y_0\,.$$

Thus, the standart form becomes:

$$f_x(x_0,y_0)\cdot x+f_y(x_0,y_0)\cdot y-z+D=0\,.$$

Ex: Find an equation for the tangent plane to the surface $z=e^x\cdot\sin y$ at the point $P_0(0,\frac{\pi}{2},1)$.

$$z=f(x,y)=e^x\sin y$$
$$\Rightarrow\ f_x(x,y)=\frac{\partial}{\partial x}\left(e^x\sin y\right)=e^x\sin y\,,$$
$$\Rightarrow\ f_y(x,y)=\frac{\partial}{\partial y}\left(e^x\sin y\right)=e^x\cos y\,.$$

At the point $P_0(0,\frac{\pi}{2},1)$: $\left((x_0,y_0,z_0)=(0,\frac{\pi}{2},1)\right)$

$$\Rightarrow f_x(0,\tfrac{\pi}{2})=e^0\sin(\tfrac{\pi}{2})=1.1=1\,,$$
$$\Rightarrow f_y(0,\tfrac{\pi}{2})=e^0\cos(\tfrac{\pi}{2})=1.0=0\,.$$

$$z-z_0=f_x\cdot(x-x_0)+f_y\cdot(y-y_0)$$
$$\Rightarrow\ z-1=1.(x-0)+0.(y-\tfrac{\pi}{2})$$
$$\Rightarrow\ z-1=x\quad\Rightarrow\ z=x+1\,.$$

Incremental Approximation

- It was stated that the tangent line to the curve $y=f(x)$ at the point $P(x_0,y_0)$ is considered the best local approximation near P. If f is differentiable at $x=x_0$ and Δx is sufficiently small, it can be written as

$$\Delta y = f(x_0+\Delta x) - f(x) \approx f'(x_0).\Delta x$$

or

$$\boxed{f(x_0+\Delta x) \approx f(x_0) + f'(x_0).\Delta x.}$$

- Likewise, the target plane at $P(x_0,y_0,z_0)$ is recognized as the best fit to the surface $z=f(x,y)$, with an analogous incremental approximation formula used.

If $f(x,y)$ and its partial derivatives f_x and f_y are defined and continuous at $P(x_0,y_0)$ in an open region, then

$$\Delta f = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) \approx f_x(x_0,y_0).\Delta x + f_y(x_0,y_0).\Delta y$$

or

$$\boxed{f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0,y_0) + f_x(x_0,y_0).\Delta x + f_y(x_0,y_0).\Delta y}.$$

Total Differential

For $y=f(x)$, $dy = f'(x)\,dx$,

for $z=f(x,y)$, $dz = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy = f_x(x,y)\,dx + f_y(x,y)\,dy$,

for $w=f(x,y,z)$,

$$dw = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$dw = f_x(x,y,z)\,dx + f_y(x,y,z)\,dy + f_z(x,y,z)\,dz.$$

Ex. Find the total differential of the functions:

$$h(x,y) = e^{x^2-2y^2} \cdot \ln(3x^2+2y^3),$$

$$k(x,y,z) = x^3 - y^4 + z^5.$$

$$dh = h_x\,dx + h_y\,dy$$

$$\Rightarrow dh = \left[2x e^{x^2-2y^2} \cdot \ln(3x^2+2y^3) + e^{x^2-2y^2} \cdot \frac{6x}{3x^2+2y^3}\right] dx$$
$$+ \left[-4y e^{x^2-2y^2} \cdot \ln(3x^2+2y^3) + e^{x^2-2y^2} \cdot \frac{6y^2}{3x^2+2y^3}\right] dy.$$

$$dk = k_x\,dx + k_y\,dy + k_z\,dz$$

$$\Rightarrow dk = 3x^2\,dx - 4y^3\,dy + 5z^4\,dz.$$

Differentiability

If $y=f(x)$ is differentiable at point x_0, its increment is determined by

$$\Delta y = f(x_0+\Delta x) - f(x_0) = f'(x_0)\,\Delta x + \varepsilon\,\Delta x.$$

Here, $\varepsilon \to 0$ as $\Delta x \to 0$. Similarly,

the function $z=f(x,y)$ is differentiable at point (x_0,y_0) if the increment of f can be written as

$$\Delta z = f_x(x_0,y_0).\Delta x + f_y(x_0,y_0).\Delta y + \varepsilon_1.\Delta x + \varepsilon_2.\Delta y.$$

Here, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

Moreover, $f(x,y)$ is considered differentiable in a region R of the plane if it is differentiable at every point in R.

Ex. Let $f(x,y,z) = e^{xyz} \cdot \ln y$. Find f_x, f_y, f_z.

$$f_x = yz \cdot e^{xyz} \cdot \ln y$$

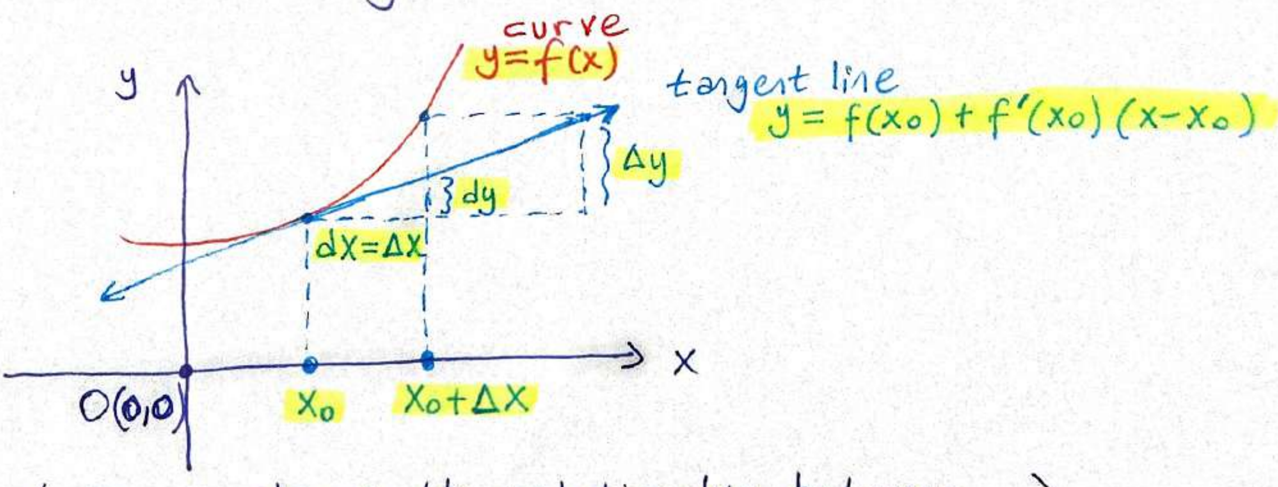
$$f_y = xz\, e^{xyz} \ln y + e^{xyz} \cdot \frac{1}{y}$$

$$f_z = xy\, e^{xyz} \ln y.$$

For f_x, consider $y = \text{const.}$, $z = \text{const.}$,
for f_y, consider $x = \text{const.}$, $z = \text{const.}$,
for f_z, consider $x = \text{const.}$, $y = \text{const.}$

We defined the differential dx for a function of one variable $y=f(x)$ as an independent variable, so dx can be given any real value. Then the differential of y is defined by

$$dy = f'(x)\,dx.$$



{ Figure shows the relationship between the increment Δy and the differential dy. }

For a differentiable function $z=f(x,y)$, we define the differentials dx and dy as independent variables. Hence, they can be given any values.

The differential dz, also called total differential is given by

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

$$\Rightarrow \quad dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy \,.$$

Sometimes the notation df is used instead of dz.

If $dx = \Delta x = x-a$ and $dy = \Delta y = y-b$ are taken, the differential of z is determined by

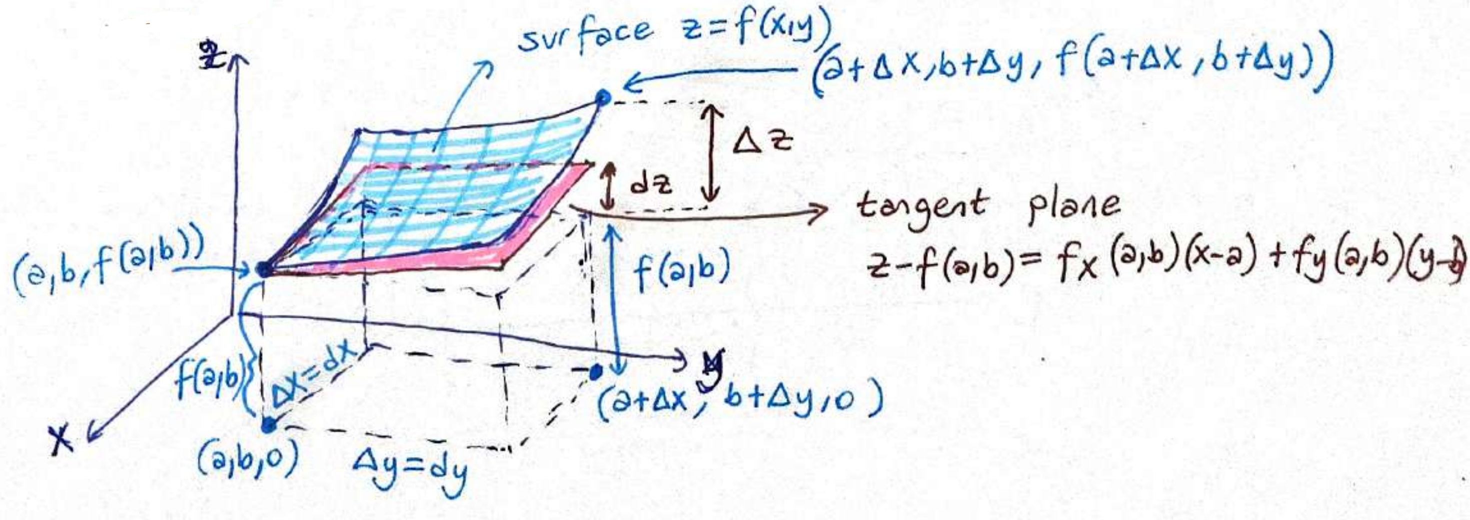
$$dz = f_x(a,b)(x-a) + f_y(a,b)(y-b)\,.$$

Thus, the linear approximation

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

with differential representation is described by

$$f(x,y) \approx f(a,b) + dz\,.$$



Ch. 11.5 Chain Rules

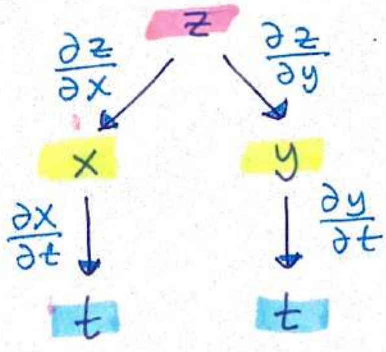
Chain Rule for One Parameter

Consider a differentiable function $f(x,y)$. If $x=x(t)$ and $y=y(t)$ are functions of the parameter t, then $z=f\big(x(t),y(t)\big)$ is a composite function.

The chain rule for its derivative with respect to t states:

If $f(x,y)$ is differentiable and $x(t)$ and $y(t)$ are differentiable functions of t, then $z=f(x,y)$ is differentiable in t, and the derivative is given by

$$\boxed{\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}}.$$



Chain Rule with tree diagram

The three diagram represents the chain rule, starting with the dependent variable z at the top and branching to independent variables x and y, which depend on parameter t. Each segment is labeled with a derivative. The chain rule derived by multiplying the derivatives along each branch and summing them, leading to:

$$\boxed{\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}}.$$

Ex. (From textbook):

Find the slope of the line that is parallel to the xz-plane and tangent to the surface $z = x\sqrt{x+y}$ at the point $P(1,3,2)$.

$$f(x,y) = x(x+y)^{1/2} \implies f_x(1,3) = ?$$

$$f_x(x,y) = 1.(x+y)^{1/2} + x\,\frac{1}{2}\cdot\frac{1}{(x+y)^{1/2}}$$

$$\implies f_x(x,y) = \frac{x}{2\sqrt{x+y}} + \sqrt{x+y}$$

$$\implies f_x(1,3) = \frac{(1)}{2\sqrt{(1)+(3)}} + \sqrt{(1)+(3)} = \frac{9}{4}.$$

Ex. Let $z = 2x^2 - 3y^3$, where $x = t^2$, $y = \frac{1}{t^3}$.

Compute $\frac{dz}{dt}$ with:

a) by first expressing z explicitly in terms of t,

b) by using the chain rule.

a) By writing $x = t^2$, $y = \frac{1}{t^3}$ $(t \neq 0)$

$$\implies z = 2x^2 - 3y^3 = 2(t^2)^2 - 3\left(\frac{1}{t^3}\right)^3 = 2t^4 - \frac{3}{t^9}$$

$$\implies \frac{dz}{dt} = 8t^3 + \frac{27}{t^{10}}$$

b) $\frac{\partial z}{\partial x} = 4x$, $\frac{\partial z}{\partial y} = -9y^2$, $\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = -\frac{3}{t^4}$

$$\implies \frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= 4x \cdot 2t + (-9y^2)\cdot\left(-\frac{3}{t^4}\right) \qquad \left(x = t^2, y = \frac{1}{t^3}\right)$$

$$= 4(t^2)\,2t + (-9)\left(\frac{1}{t^3}\right)^2\left(-\frac{3}{t^4}\right) = 8t^3 + \frac{27}{t^{10}}.$$

Ex. What value of c makes the following function continuous at $(0,0)$?

$$f(x,y)=\begin{cases} \dfrac{x^2+y^2}{x^2+y^2+1} & \text{if } (x,y)\neq(0,0) \\ c & \text{if } (x,y)=(0,0) \end{cases}$$

When $x\to 0,\ y\to 0 \Rightarrow x^2+y^2\to 0$ since $(x^2\to 0,\ y^2\to 0)$.

Then, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+y^2+1} = \frac{0}{0+1} = 0.$

For the function to be continuous at $(0,0)$, the value of $f(0,0)$, which is c, must equal the limit we just calculated. So, $c=0$. That is,

$$\Rightarrow \underbrace{\lim_{(x,y)\to(0,0)} f(x,y)}_{=0} = \underbrace{f(0,0)}_{=0} = c.$$

Ex. Let $z=\dfrac{xy^5-5x^2}{y^2+1}$. Find z_x and z_y.

Using the quotient rule, where $\boxed{u=xy^5-5x^2}$, $\boxed{v=y^2+1}$:

$$z_x=\frac{\partial z}{\partial x}=\frac{\frac{\partial u}{\partial x}\cdot v-u\cdot\frac{\partial v}{\partial x}}{v^2}=\frac{(y^5-10x)(y^2+1)-(xy^5-5x^2)\cdot 0}{(y^2+1)^2}$$

$$\Rightarrow \boxed{z_x=\frac{y^5-10x}{y^2+1}}.$$

$$\boxed{z_y}=\frac{\partial z}{\partial y}=\frac{\frac{\partial u}{\partial y}\cdot v-u\cdot\frac{\partial v}{\partial y}}{v^2}=\boxed{\frac{5xy^4\cdot(y^2+1)-(xy^5-5x^2)\,2y}{(y^2+1)^2}}.$$

To apply the chain rule for a function $f(x,y,z)$ of three variables dependent on three parameters u,v,w:

- Express x,y,z as functions of u,v,w:

$$x = x(u,v,w), \quad y = y(u,v,w), \quad z = z(u,v,w).$$

- The total derivatives are given by

$$\boxed{\begin{aligned} \frac{df}{du} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u}, \\ \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial v}, \\ \frac{\partial f}{\partial w} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial w} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial w}. \end{aligned}}$$

Ex. Let $f(x,y,z) = \sin(x\cdot y) + e^z + \ln(x+z)$.

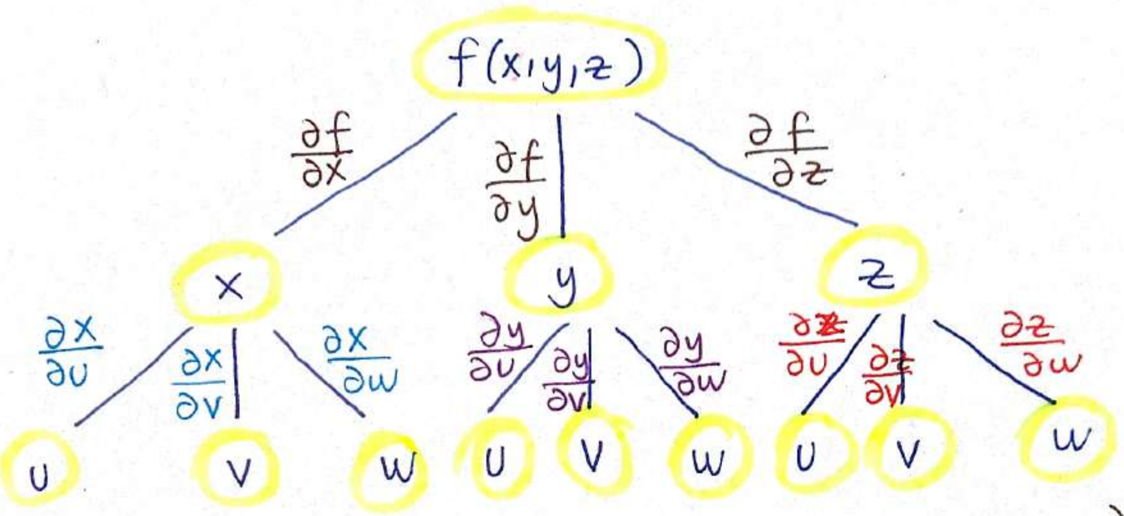
Find $\boxed{\frac{\partial f}{\partial v}}$. Here, $x = u^2+v$, $y = \sin w$, $z = e^u + v^2$.

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= y\cos(xy) + \frac{1}{x+z}, \\ \frac{\partial f}{\partial y} &= x\cos(xy), \\ \frac{\partial f}{\partial z} &= e^z + \frac{1}{x+z}. \end{aligned}\right\} \quad \begin{aligned} \frac{\partial x}{\partial v} &= 1, \\ \frac{\partial y}{\partial v} &= 0, \\ \frac{\partial z}{\partial v} &= 2v. \end{aligned}$$

Then, $\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial v}$

$$\Rightarrow \frac{\partial f}{\partial v} = \left(y\cos(xy) + \frac{1}{x+z}\right)\cdot 1 + \left[x\cos(xy)\right]\cdot 0 + \left(e^z + \frac{1}{x+z}\right)\cdot 2v$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial v} = y\cos(xy) + \frac{1}{x+z} + 2v\left(e^z + \frac{1}{x+z}\right).}$$



Chain Rule (with tree diagram) of three variables (x,y,z) with three parameters (u,v,w).

Implicit Function Theorem:

Let F be a function defined on a disk containing (a,b), with $F(a,b)=0$. If F_x and F_y are continuous in the disk and $F_y(a,b) \neq 0$, then there exists an interval I around a and a unique function $y=y(x)$ defined on I, such that $y(a)=b$ and $F(x,y(x))=0$ for all $x \in I$.

The derivative of $y(x)$ is determined by

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}.$$

Ex. If $\cos(x+y) - \sin(x-y) = x$, where y is a differentiable function of x, then find $\frac{dy}{dx}$.

$$F(x,y) = \cos(x+y) - \sin(x-y) - x$$

$$\Rightarrow F_x = -\sin(x+y) - \cos(x-y) - 1,$$

$$\Rightarrow F_y = -\sin(x+y) - (-1)\cos(x-y)$$

$$\Rightarrow \boxed{\frac{dy}{dx}} = -\frac{-\sin(x+y) - \cos(x-y) - 1}{-\sin(x+y) + \cos(x-y)} = \boxed{\frac{\sin(x+y) + \cos(x-y) + 1}{-\sin(x+y) + \cos(x-y)}}.$$

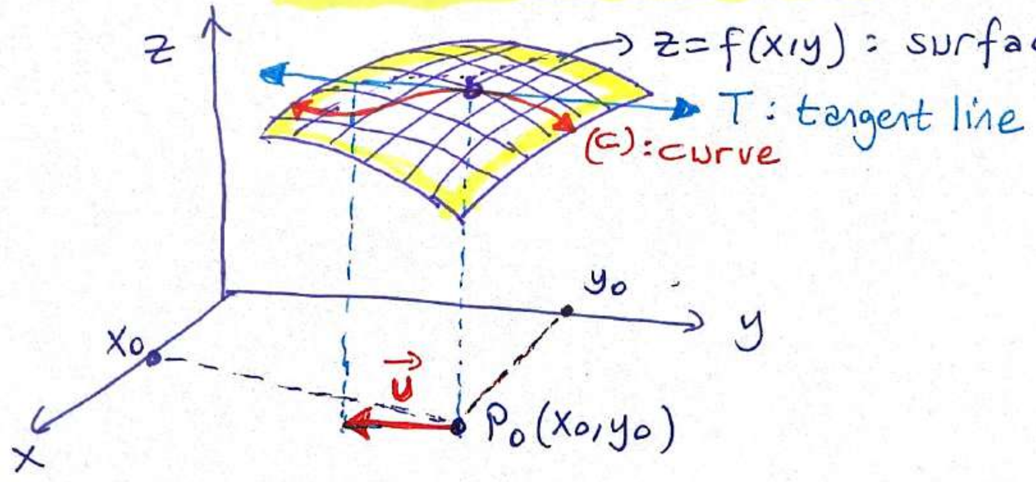
Ch. 11.6 Directional Derivatives and the Gradient

The Directional Derivative

The directional derivative extends the concept of slope from curves to surfaces.

For a surface $z=f(x,y)$, the slope at a point $P_0(x_0,y_0)$ depends on the direction of measurement, which is specified using vectors. The slope parallel to the xz-plane is given by the partial derivative $f_x(x_0,y_0)$, while $f_y(x_0,y_0)$ gives the slope in the yz-plane.

For an arbitrary direction, the slope is measured using a unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j} = \langle u_1, u_2 \rangle$.



The directional derivative of a function f at $P_0(x_0,y_0)$ in the direction of a unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is defined as

$$\boxed{D_{\vec{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h.u_1,\, y_0+h.u_2) - f(x_0,y_0)}{h}}$$

as long as the limit exists.

At the point $P_0(x_0, y_0)$, there are infinitely many directional derivatives for the surface $z = f(x, y)$.

The partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ represent the derivatives in the x-direction and y-direction, respectively. Specifically, f_x is obtained when $\vec{u}$ is along $\vec{i}$, and f_y is obtained when $\vec{u}$ is along $\vec{j}$.

If $f(x, y)$ is differentiable at $P_0(x_0, y_0)$, the directional derivative in the direction of the unit vector $\vec{u} = u_1 \vec{i} + u_2 \vec{j}$ is determined by

$$\boxed{D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)\, u_1 + f_y(x_0, y_0)\, u_2}.$$

Ex. Find the directional derivative of

$$g(x, y) = 5 + x^2 - 4y^2$$

at $Q(2, 1)$ in the direction of the unit vector $\vec{v} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$.

$$g_x(x, y) = 2x \implies g_x(2, 1) = 2.(2) = 4,$$

$$g_y(x, y) = -8y \implies g_y(2, 1) = -8.(1) = -8.$$

$$\implies D_{\vec{v}}\, g(2, 1) = g_x(2, 1).v_1 + g_y(2, 1).v_2$$

$$= 4 \cdot \left(\frac{3}{5}\right) + (-8) \cdot \left(\frac{4}{5}\right)$$

$$= \frac{12}{5} - \frac{32}{5}$$

$$\implies D_{\vec{v}}\, g(2, 1) = -4.$$

The Gradient

The directional derivative $D_u f(x,y)$ can be expressed using the gradient of a differentiable function f with partial derivatives f_x and f_y as

$$\boxed{\nabla f(x,y) = f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j}\,.} \qquad \{\nabla : \text{del}\}$$

At $P_0(x_0, y_0)$, the gradient is represented by ∇f_0.

$$\{\nabla f = \text{grad}(f)\}$$

Ex. Find $\nabla f(x,y)$ for the function $f(x,y) = \dfrac{x^3y^2}{1+x^2+y^2}$.

Using the quotient rule :

$$f_x(x,y) = \frac{(3x^2y^2)(1+x^2+y^2) - (2x)(x^3y^2)}{(1+x^2+y^2)^2}$$

$$\Rightarrow f_x(x,y) = \frac{x^2y^2[3(1+x^2+y^2) - 2x^2]}{(1+x^2+y^2)^2}$$

$$\Rightarrow f_x(x,y) = \frac{x^2y^2(3+x^2+3y^2)}{(1+x^2+y^2)^2}\,, \quad \text{and}$$

$$f_y(x,y) = \frac{(2x^3y)(1+x^2+y^2) - (x^3y^2)(2y)}{(1+x^2+y^2)^2}$$

$$\Rightarrow f_y(x,y) = \frac{x^3y[2(1+x^2+y^2) - 2y^2]}{(1+x^2+y^2)^2} = \frac{x^3y(2+2x^2)}{(1+x^2+y^2)^2}\,.$$

Then, $\nabla f(x,y) = f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j}$

$$\Rightarrow \boxed{\nabla f(x,y) = \frac{x^2y^2(3+x^2+3y^2)}{(1+x^2+y^2)^2}\,\vec{i} + \frac{2x^3y(1+x^2)}{(1+x^2+y^2)^2}\,\vec{j}}$$

The directional derivative of a differentiable function f at the point $P_0(x_0, y_0)$ in the direction of the unit vector $\vec{u}$ is determined by

$$\boxed{D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}},$$

where $\nabla f(x_0, y_0)$ is the gradient of f at P_0.

$$\begin{aligned} D_{\vec{u}} f(x_0, y_0) &= \nabla f(x_0, y_0) \cdot \vec{u} = \nabla f_0 \cdot \vec{u} \\ &= \left[f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j} \right] \cdot (u_1 \vec{i} + u_2 \vec{j}) \\ &= f_x(x_0, y_0) u_1 + f_y(x_0, y_0) u_2 . \end{aligned}$$

Ex. Find the directional derivative of the function $g(x,y) = e^{x^2+y^3}$ at the point $Q_0(2,1)$ in the direction of the vector $\vec{w} = 3\vec{i} + 4\vec{j}$.

$$g_x(x,y) = 2x \cdot e^{x^2+y^3} \Rightarrow g_x(2,1) = 2 \cdot (2) \cdot e^{(2)^2+(1)^3} = 4e^5,$$

$$g_y(x,y) = 3y^2 e^{x^2+y^3} \Rightarrow g_y(2,1) = 3(1)^2 e^{(2)^2+(1)^3} = 3e^5.$$

$$\Rightarrow \nabla g(2,1) = 4e^5 \vec{i} + 3e^5 \vec{j} = \langle 4e^5, 3e^5 \rangle .$$

$$\vec{w} = \langle 3, 4 \rangle \Rightarrow \|\vec{w}\| = \sqrt{3^2+4^2} = 5 .$$

$$\vec{u} = \frac{\vec{w}}{\|\vec{w}\|} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle . \text{ Then,}$$

unit vector in the direction of $\vec{w}$

$$\begin{aligned} D_{\vec{u}} g(2,1) &= \nabla g(2,1) \cdot \vec{u} \\ &= \langle 4e^5, 3e^5 \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \\ &= 4e^5 \cdot \frac{3}{5} + 3e^5 \cdot \frac{4}{5} = \frac{24e^5}{5} . \end{aligned}$$

Ex. Let $\boxed{f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x-y} & \text{if } x-y \neq 0 \\ h(x,y) & \text{if } x-y=0 \end{cases}}$.

Find the function $h(x,y)$ such that $f(x,y)$ is continuous at all points in the plane.

The function $f(x,y) = \dfrac{x^3-y^3}{x-y}$ for $x \neq y$.

$x^3-y^3 = (x-y)(x^2+xy+y^2)$. Using this factorization

$$f(x,y) = \frac{(x-y)(x^2+xy+y^2)}{x-y} = x^2+xy+y^2 \quad \text{for } x \neq y.$$

$$\Rightarrow \lim_{x \to y} f(x,y) = \lim_{x\to y} (x^2+xy+y^2) = (y)^2 + (y)\cdot y + y^2 = 3y^2.$$

$\left\{ \begin{array}{l} \text{Means} \\ x-y \to 0 \end{array} \right\}$ The function $f(x,y)$ continuous everywhere if we define $h(x,y) = 3y^2$.

Therefore, the final form is determined by

$$\boxed{f(x,y) = \begin{cases} x^2+xy+y^2 & \text{if } x \neq y \\ 3y^2 & \text{if } x = y \end{cases}}.$$

Ex. Let $z = z(x,y)$ be a continuous function of x and y, implicitly defined by $x^2z + y^3z^2 + x^3yz^3 = 6$.

Find $\dfrac{\partial z}{\partial y}$, where it is defined.

$$\frac{\partial}{\partial y}(x^2z + y^3z^2 + x^3yz^3) = \frac{\partial}{\partial y}(6)$$

$$\Rightarrow \frac{\partial}{\partial y}(x^2z) + \frac{\partial}{\partial y}(y^3z^2) + \frac{\partial}{\partial y}(x^3yz^3) = 0$$

$$\Rightarrow \left(x^2\frac{\partial z}{\partial y}\right) + \left(3y^2z^2 + y^3 \cdot 2z\frac{\partial z}{\partial y}\right) + \left(x^3z^3 + 3x^3yz^2\frac{\partial z}{\partial y}\right) = 0$$

$$\Rightarrow \frac{\partial z}{\partial y}(x^2 + 2y^3z + 3x^3yz^2) = -3y^2z^2 - x^3z^3 \Rightarrow \frac{\partial z}{\partial y} = \frac{-3y^2z^2 - x^3z^3}{x^2+2y^3z+3x^3yz^2}$$

Ex. Find the equation of the plane tangent to the surface $z = 4 - 3x^2 + y^3$ at the point $(x,y) = (2,1)$.

The general form of the equation of a tangent plane to the surface $z = f(x,y)$ at the point (x_0, y_0, z_0) is given by

$$\boxed{z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)},$$

where f_x and f_y are the partial derivatives of $f(x,y)$.

$$f_x(x,y) = \frac{\partial}{\partial x}(4 - 3x^2 + y^3) = -6x,$$

$$\Rightarrow f_x(2,1) = -6.(2) = -12.$$

$$f_y(x,y) = \frac{\partial}{\partial y}(4 - 3x^2 + y^3) = 3y^2,$$

$$\Rightarrow f_y(2,1) = 3.(1)^2 = 3.$$

$$\left\{ \begin{matrix} (x_0, y_0, z_0) \\ \| \\ (2, 1, -7) \end{matrix} \right\}$$

$$z_0 = f(2,1) = 4 - 3.(2)^2 + (1)^3 = 4 - 12 + 1 = -7.$$

Then, equation of the tangent plane is determined by

$$z - (-7) = -12(x - 2) + 3(y - 1)$$

$$\Rightarrow z + 7 = -12x + 24 + 3y - 3$$

$$\Rightarrow \boxed{12x - 3y + z - 14 = 0}.$$

Ex. Consider the function $\boxed{G(x,y) = \ln(x^3y^4)}$, where $x = u^2v$ and $y = uv^3$. Compute $\boxed{\frac{\partial G}{\partial u}}$ using the chain rule, where the partial derivative exists.

$$\frac{\partial G}{\partial x} = \frac{3x^2y^4}{x^3y^4} = \frac{3}{x}, \qquad \frac{\partial x}{\partial u} = 2uv,$$

$$\frac{\partial G}{\partial y} = \frac{4x^3y^3}{x^3y^4} = \frac{4}{y}, \qquad \frac{\partial y}{\partial u} = v^3.$$

Using the chain rule :

$$\Rightarrow \boxed{\frac{\partial G}{\partial u} = \frac{\partial G}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial G}{\partial y}\cdot\frac{\partial y}{\partial u}}$$

$$\Rightarrow \frac{\partial G}{\partial u} = \frac{3}{x}\cdot 2uv + \frac{4}{y}\cdot v^3$$

$$\Rightarrow \frac{\partial G}{\partial u} = \frac{3}{u^2v}\cdot 2uv + \frac{4}{uv^3}\cdot v^3$$

Since $x = u^2v$, $y = uv^3$

$$\Rightarrow \frac{\partial G}{\partial u} = \frac{6}{u} + \frac{4}{u}$$

$$\Rightarrow \frac{\partial G}{\partial u} = \frac{10}{u}.$$

Ex. Find $\frac{\partial G}{\partial v}$ for above function.

Ex. Let $f(x,y) = x^3y^2$, and let the point $P = (2,3)$. Calculate the directional derivative of f at P in the direction of the vector $\vec{v} = 4\vec{i} - 3\vec{j}$.

The gradient of f is determined by

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

$$\Rightarrow \nabla f = \langle 3x^2y^2, 2x^3y \rangle$$

$$\Rightarrow \nabla f(2,3) = \langle 3(2)^2 \cdot (3)^2, \ 2 \cdot (2)^3 \cdot (3) \rangle$$

$$\Rightarrow \nabla f(2,3) = \langle 108, 48 \rangle$$

$$\|\vec{v}\| = \sqrt{4^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5.$$

The unit vector is given by

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 4, -3 \rangle}{5} = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle.$$

Then, directional derivative is described by

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}.$$

$$\Rightarrow D_{\vec{u}} f = \langle 108, 48 \rangle \cdot \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$$

$$\Rightarrow D_{\vec{u}} f = 108 \cdot \frac{4}{5} + 48 \cdot \left(-\frac{3}{5}\right)$$

$$\Rightarrow D_{\vec{u}} f = \frac{432}{5} - \frac{144}{5}$$

$$\Rightarrow \boxed{D_{\vec{u}} f = \frac{288}{5}}.$$ $\left\{ \text{directional derivative of } f \text{ at } (2,3). \right\}$

Ex. Find the maximum rate of change of the function $g(x,y) = x^4 + 3xy^2 + 2y^3$ at the point $Q(1,2)$, and identify the direction in which this maximum rate occurs.

The gradient ∇g is described by

$$\nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right\rangle \quad \text{(gradient vector)}$$

$$\Rightarrow \nabla g = \langle 4x^3 + 3y^2, 6xy + 6y^2 \rangle .$$

The gradient at $Q(1,2)$:

$$\nabla g(1,2) = \langle 4\cdot(1)^3 + 3(2)^2, 6(1)(2) + 6(2)^2 \rangle$$

$$\Rightarrow \nabla g(1,2) = \langle 16, 36 \rangle .$$

The maximum rate of change of the function at a point is given by the magnitude of the gradient vector:

$$\|\nabla g(1,2)\| = \sqrt{16^2 + 36^2} = \sqrt{256 + 1296} = \sqrt{1552}$$

$$\Rightarrow \|\nabla g(1,2)\| = 4\sqrt{97} \approx 39.2 \quad \text{(maximum rate of change)}$$

The direction of the maximum rate of change is given by the unit vector in the direction of $\nabla g(1,2)$:

$$\vec{u} = \frac{\nabla g(1,2)}{\|\nabla g(1,2)\|} = \frac{\langle 16, 36 \rangle}{4\sqrt{97}}$$

$$\Rightarrow \vec{u} = \left\langle \frac{4}{\sqrt{97}}, \frac{9}{\sqrt{97}} \right\rangle \quad \text{(direction)}$$

Ex. Find the equation of the tangent plane to the surface defined by

$$H(x,y,z) = 3x^2+2y^2-z+5 = 0$$

at the point $(1,2,4)$.

$$\nabla H = \left< \frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, \frac{\partial H}{\partial z} \right> = \left< 6x, 4y, -1 \right>.$$

$$\Rightarrow \nabla H(1,2,4) = \left< 6.(1), 4.(2), -1 \right> = \left< 6, 8, -1 \right>.$$

Then, the equation of the tangent plane at a point (x_0, y_0, z_0) is given by

$$\nabla H \cdot \left< x-x_0, y-y_0, z-z_0 \right> = 0$$

$$\Rightarrow \nabla H(1,2,4) \cdot \left< x-1, y-2, z-4 \right> = 0$$

$$\Rightarrow \left< 6,8,-1 \right> \cdot \left< x-1, y-2, z-4 \right> = 0$$

$$\Rightarrow 6(x-1) + 8(y-2) - (z-4) = 0$$

$$\Rightarrow 6x - 6 + 8y - 16 - z + 4 = 0$$

$$\Rightarrow \underline{6x + 8y - z - 18 = 0}. \quad \text{(tangent plane equation)}$$

Ex. Solve the above, taking

$$F(x,y,z) = 2x^2 - 3y^2 + z = 0$$

at $(-1, 3, 5)$.

The chain rule can be used to obtain a more complete description of the implicit differentiation method.

Suppose that an equation $\boxed{F(x,y)=0}$ defined y as an implicit differentiable function of x, that is, $y=f(x)$ and for all x in the domain of f, $F(x,f(x))=0$. If F is differentiable we can use chain rule to differential both sides of the equation $F(x,y)=0$ with respect to x. Since both x and y are functions of x,

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0.$$

If $\frac{dy}{dx}\neq 0$ then $\frac{dx}{dx}=1$. So, solving for $\frac{dy}{dx}$ we get

$$\boxed{\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=-\frac{F_x}{F_y}}.$$

Suppose that an equation $\boxed{F(x,y,f(x,y))=0}$ defines z implicitly as $z=f(x,y)$. This implies that $F(x,y,f(x,y))=0$ for all (x,y) in the domain of f. If F and f are differentiable, we can differentiate the equation $F(x,y,z)=0$ using the chain rule:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0.$$

However,

$$\frac{\partial}{\partial x}(x)=1 \quad\text{and}\quad \frac{\partial}{\partial x}(y)=0,$$

then

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 .$$

If $\frac{\partial F}{\partial z} \neq 0$, we obtain

$$\boxed{\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z}} .$$

Similarly, we have

$$\boxed{\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}} .$$

Ex. Let $x^4+y^4+z^4-3xyz=2$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{4x^3-3yz}{4z^3-3xy}},$$

$$\boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{4y^3-3xz}{4z^3-3xy}},$$

where $F(x,y,z) = x^4+y^4+z^4-3xyz-2$.

Ex. Let $x^5 - x^2y^3z + xy^2z^3 = xyz+1$.

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

For a function of three variables $f(x,y,z)$, the gradient denoted by ∇f or $\mathrm{grad}\, f$ is a vector

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$$

or briefly

$$\boxed{\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}}.$$

As a result, as in functions of two variables, the directional derivatives of three variables is given by

$$\boxed{D_{\vec{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \vec{u}}.$$

Ex. Let $f(x,y,z) = z.\sin(xy)$.

a) Find ∇f,

b) Find the directional of f at $(1,2,0)$ in the direction of $\vec{v} = \vec{i} + 2\vec{j} + \vec{k}$.

a) $\nabla f = \langle f_x, f_y, f_z \rangle = \langle yz\cos(xy), xz\cos(xy), \sin(xy) \rangle$

b) $\nabla f(1,2,0) = \langle (2).(0).\cos(1.2), (1).(0).\cos(1.2), \sin(1.2) \rangle$

$\nabla f(1,2,0) = \langle 0, 0, \sin 2 \rangle$ $\quad \{\sin 2 \approx 0.9\}$

$= \langle 0, 0, 0.9 \rangle$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1,2,1 \rangle}{\sqrt{1^2+2^2+1^2}} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle.$$

$$\Rightarrow D_{\vec{u}} f(1,2,0) = \nabla f(1,2,0) \cdot \vec{u} = \langle 0,0,0.9 \rangle \cdot \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

$$\Rightarrow D_{\vec{u}} f(1,2,0) = \frac{0.9}{\sqrt{6}} = \frac{9}{10\sqrt{6}}.$$

Some Properties of the Gradient

Let h and k be differentiable functions. Then,

- $\nabla c = \vec{0}$, $\forall c \in \mathbb{R}$ (constant)
- $\nabla(ah + bk) = a.\nabla h + b.\nabla k$, $\forall a, b \in \mathbb{R}$ (linearity)
- $\nabla(h.k) = (\nabla h).k + h.(\nabla k)$ (product)
- $\nabla\left(\frac{h}{k}\right) = \frac{(\nabla h).k - h.(\nabla k)}{k^2}$ (quotient)
- $\nabla(h^n) = n.h^{n-1}.(\nabla h)$ (power)

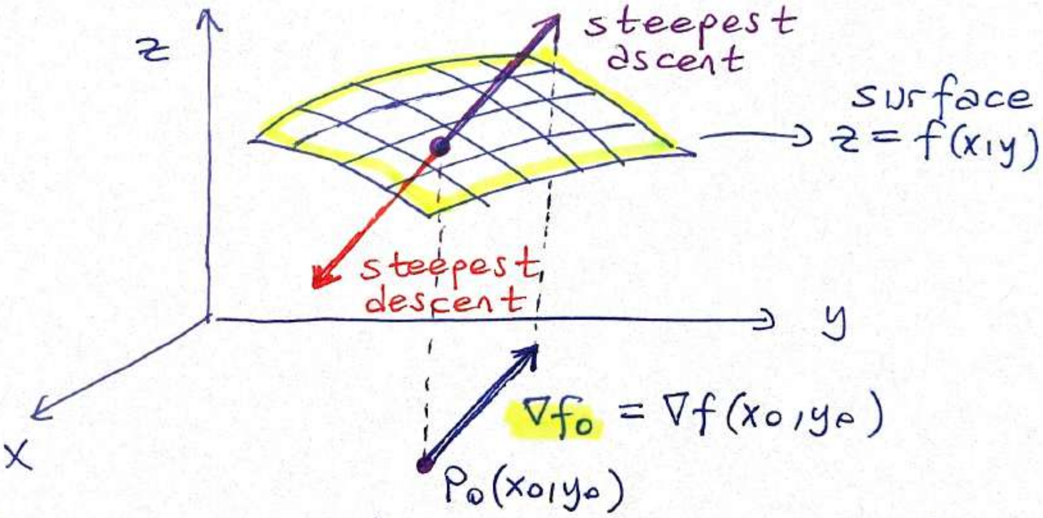
Proof of power rule :

$$\nabla(h^n) = (h^n)_x \vec{i} + (h^n)_y \vec{j}$$
$$= n h^{n-1} h_x \vec{i} + n.h^{n-1} h_y \vec{j}$$
$$= n h^{n-1} (h_x \vec{i} + h_y \vec{j})$$
$$\Rightarrow \nabla(h^n) = n h^{n-1}.(\nabla h).$$

Proof of constant rule :

$$\nabla c = (c)_x \cdot \vec{i} + (c)_y \cdot \vec{j}$$
$$= 0.\vec{i} + 0.\vec{j}$$
$$= \langle 0, 0 \rangle \quad \text{(zero vector in } \mathbb{R}^2\text{)}$$
$$\Rightarrow \nabla c = \vec{0} \text{ in } \mathbb{R}^2.$$

The Maximal Property of the Gradient states that the greatest rate of change of a function at a point occurs in the direction of the gradient. The steepest descent is in the opposite direction of the gradient. For example, if $z=f(x,y)$ represents the height of a mountain, a hiker's steepest descent follows the direction opposite to $\nabla f(x,y)$.



$$\begin{pmatrix}\text{Magnitude of} \\ \text{steepest ascent} \\ \text{tangent vector}\end{pmatrix} = \begin{pmatrix}\text{Magnitude of} \\ \text{steepest descent} \\ \text{tangent vector}\end{pmatrix} = \|\nabla f_0\|$$

Maximal Direction Property of the Gradient

If f is a differentiable at P_0 and $\nabla f_0 \neq 0$, then :

i.) The largest value of the directional derivative $D_{\vec{u}}f$ at P_0 is $\|\nabla f_0\|$, occuring when $\vec{u}$ aligns with ∇f_0.

ii.) The smallest value of $D_{\vec{u}}f$ at P_0 is $-\|\nabla f_0\|$, occuring when $\vec{u}$ aligns with $-\nabla f_0$.

Ex. In what direction is the function defined by $g(x,y) = 2xy + y^3 - 3x^2$ increasing most rapidly at the point $Q_0(1,2)$, and what is the maximum rate of increase? In what direction is g decreasing most rapidly?

$$\nabla g = \langle g_x, g_y \rangle = \langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \rangle$$

$$\Rightarrow \nabla g = \langle 2y - 6x, 2x + 3y^2 \rangle$$

$$\nabla g(1,2) = \langle 2.(2) - 6(1), 2.(1) + 3(2)^2 \rangle = \langle 4-6, 2+12 \rangle$$

$$\Rightarrow \nabla g(1,2) = \underline{\langle -2, 14 \rangle} : \left\{ \text{direction of the maximum increase} \right\}$$

$$\|\nabla g(1,2)\| = \sqrt{(-2)^2 + (14)^2} = \sqrt{4+196} = \sqrt{200} \approx 14.14$$

maximum rate of increase

$$-\nabla g(1,2) = \langle 2, -14 \rangle$$

direction of maximum decrease

The maximal direction property of the gradient indicates that at a point P_0, the function f increases most rapidly in the direction of the gradient ∇f_0 and degreases most rapidly in the opposite direction, $-\nabla f_0$.

The concepts of directional derivatives and gradients extends to functions of three or more variables. For $f(x,y,z)$ the gradient is defined as

$$\boxed{\nabla f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}},$$

and the directional derivative $D_{\vec{u}} f$ at $P_0(x_0,y_0,z_0)$ in the direction of unit vector $\vec{u}$ is

$$\boxed{D_{\vec{u}} f = \nabla f_0 \bullet \vec{u}}.$$

The properties of the gradient and the maximal direction property remain valid for higher-dimensional functions.

Ex. Let $f(x,y,z) = x^2 y \cos z$.

a) Find the gradient ∇f at the point $P_0(2,-1,\frac{\pi}{3})$.

b) compute the directional derivative of f at P_0 in the direction of the vector $\vec{v} = \vec{i} - 2\vec{j} + \vec{k}$.

a) $\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2xy\cos z, x^2 \cos z, -x^2 y \sin z \rangle$.

$\nabla f_0 = \langle 2(2)(-1)\cos\frac{\pi}{3}, (2)^2 \cos\frac{\pi}{3}, -(2)^2(-1)\cdot\sin\frac{\pi}{3} \rangle$

$\nabla f_0 = \langle -4\cdot\frac{1}{2}, 4\cdot\frac{1}{2}, 4\cdot\frac{\sqrt{3}}{2} \rangle$

$\nabla f_0 = \langle -2, 2, 2\sqrt{3} \rangle$.

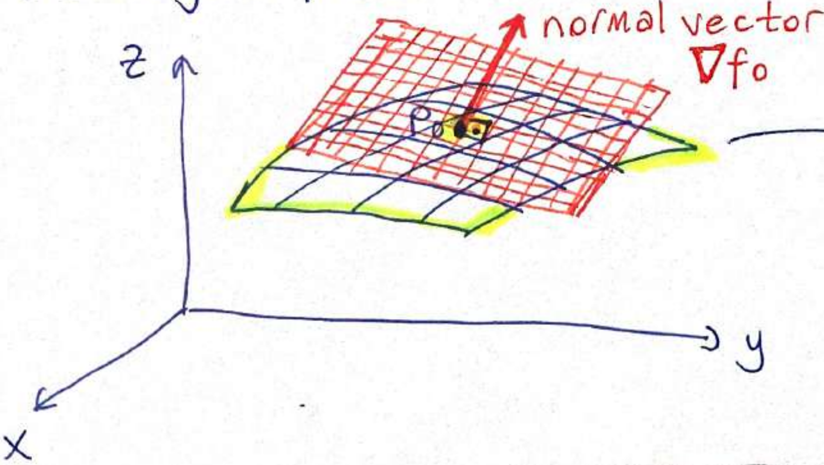
b) $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{\langle 1,-2,1 \rangle}{\sqrt{1^2+(-2)^2+1^2}} = \langle \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \rangle$ (unit vector)

$\Rightarrow D_{\vec{u}} f = \nabla f_0 \bullet \vec{u} = \langle -2, 2, 2\sqrt{3} \rangle \bullet \langle \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \rangle$

$\Rightarrow D_{\vec{u}} f = (-2)\frac{1}{\sqrt{6}} + (2)\left(-\frac{2}{\sqrt{6}}\right) + 2\sqrt{3}\cdot\frac{1}{\sqrt{6}} = \dfrac{-6+2\sqrt{3}}{\sqrt{6}}$.

The Normal Property of the Gradient indicates that if S is a level surface defined by $f(x,y,z) = \mathbf{C}$ ($\mathbf{C}$ is any constant), then at any point $P_0(x_0, y_0, z_0)$ on S, the gradient ∇f_0 is normal (orthogonal) to the tangent plane at P_0.

That is, the gradient ∇f_0 at each point P_0 on the surface $f(x,y,z) = C$ is orthogonal to the tangent vector $\vec{T} = \frac{d\vec{r}}{dt}$ of any curve $\vec{r}(t)$ on the surface passing through P_0. All tangent vectors lie in a plane through P_0 with the normal vector $\vec{N} = \nabla f_0$, which defines the tangent plane at P_0.



$\{ \nabla$: del = nabla operator $\}$

Ex. Find a vector that is normal to the level surface defined by $x^2 - 3y^2 + 4yz + z^2 = 10$ at $P_0(2,1,1)$.

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2x, -6y + 4z, 4y + 2z \rangle$$

$$\nabla f_0 = \langle 2(2), -6(1) + 4\cdot(1), 4\cdot(1) + 2(1) \rangle = \langle 4, -2, 6 \rangle.$$

$$\Rightarrow \vec{N} = \nabla f_0 = \langle 4, -2, 6 \rangle \quad \text{(normal vector)}$$

Normal Line and Tangent Plane with Formulas

Normal lines and the tangent planes to a surface extend the concepts from $\mathbb{R}^2$ to $\mathbb{R}^3$. For a surface S with a normal vector $\vec{N}$ at point P_0, the normal line is the line through P_0 with normal vector $\vec{N}$.

If the surface S has a nonzero normal vector $\vec{N}$ at P_0, then :

- The line through P_0 parallel to $\vec{N}$ is the normal line to S at P_0.
- The plane through P_0 with normal vector $\vec{N}$ is the tangent plane to S at P_0.

Let S be a surface defined by the equation $F(x,y,z)=C$, and let $P_0(x_0,y_0,z_0)$ be a point on S where F is differentiable and $\nabla F \neq 0$. Then,

$$\boxed{F_x(x_0,y_0,z_0)\cdot(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0}$$

(tangent plane equation at P_0)

and

$$\boxed{\begin{aligned} x &= x_0 + F_x(x_0,y_0,z_0).t \;, \\ y &= y_0 + F_y(x_0,y_0,z_0).t \;, \\ z &= z_0 + F_z(x_0,y_0,z_0).t \;. \end{aligned}}$$

{normal line parametric equations at P_0.}

Ex. Find the equations for the tangent plane and the normal line to the surface defined by the paraboloid $z = x^2 + y^2$ at the point where $x = 2$, $y = 3$.

z at the point $(2,3)$:

$$z = 2^2 + 3^2 = 4 + 9 = 13 .$$

$F(x,y,z) = z - x^2 - y^2$. (We set $F(x,y,z) = 0$ for paraboloid)

$$\nabla F = \langle F_x, F_y, F_z \rangle = \langle -2x, -2y, 1 \rangle$$

$$\nabla F(2,3,13) = \nabla F_0 = \langle -2(2), -2(3), 1 \rangle = \langle -4, -6, 1 \rangle .$$

The equation of the tangent plane:

$$\boxed{F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0}$$

$\Rightarrow \quad -4.(x-2) + (-6).(y-3) + 1.(z-13) = 0$

$\Rightarrow \quad -4x + 8 - 6y + 18 + z - 13 = 0$

$\Rightarrow \quad -4x - 6y + z + 13 = 0$, or in standart form:

$\Rightarrow \quad 4x + 6y - z - 13 = 0 .$

Parametric equations of the normal line:

$$\boxed{\begin{aligned} x &= x_0 + F_x(x_0,y_0,z_0).t = 2 - 4.t , \\ y &= y_0 + F_y(x_0,y_0,z_0).t = 3 - 6.t , \\ z &= z_0 + F_z(x_0,y_0,z_0).t = 13 + 1.t . \end{aligned}}$$

$\Rightarrow \quad x = 2 - 4t , \; y = 3 - 6t , \; z = 13 + t .$

Ch. 11.7 Extrema of Functions of Two Variables

Absolute extrema refer to the highest and lowest values a function can take in its domain. A function has an "absolute maximum" at (x_0, y_0) if

$$\boxed{f(x_0, y_0) \geq f(x, y)}$$

for all points (x, y) in the domain. Similarly, it has an "absolute minimum" at (x_0, y_0) if

$$\boxed{f(x_0, y_0) \leq f(x, y)}$$

for all points in the domain.

Relative extrema occur when a function reaches a maximum or minimum value within a spesific region.

A function has a "relative maximum" at (x_0, y_0) if

$$\boxed{f(x, y) \leq f(x_0, y_0)}$$

for all points (x, y) in a surrounding open disk. It has a "relative minimum" if

$$\boxed{f(x, y) \geq f(x_0, y_0)}$$

in the same region. Together these are called relative extrema.

Relative extrema are found where $f'(x) = 0$ or $f'(x)$ does not exist for one-variable functions.

Similarly, for two-variables:

If a function $z = f(x, y)$ has a relative extremum (max. or min.) at $P_0(x_0, y_0)$ and f_x and f_y exist at P_0, then

$$\boxed{f_x(x_0, y_0) = f_y(x_0, y_0) = 0.}$$

A critical point of a function $f(x,y)$ defined on an open set D is a point (x_0,y_0) in D where one of the following conditions holds:

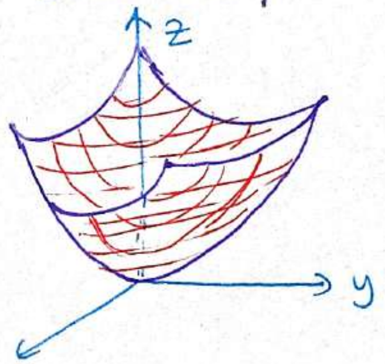
- $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$
- At least f_x or f_y does not exist on (x_0,y_0).

Ex. Analyze the critical point $(0,0)$ for the following quadric surfaces:

1) $z = 2x^2 + 3y^2$,
2) $z + 2x^2 + 3y^2 = 4$
3) $z = x^2 - 2y^2$.

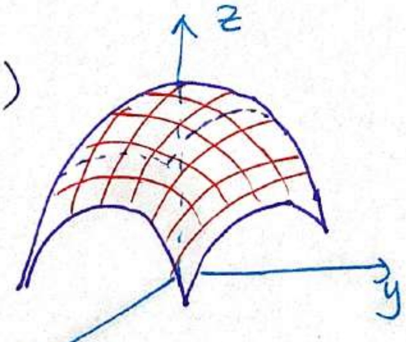
Then, determine the nature of the critical point for each surface.

1)



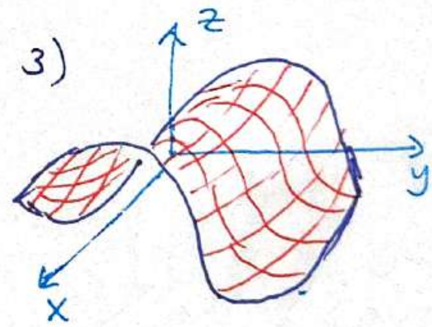
upward-opening paraboloid

2)



downward-opening paraboloid

3)



hyperbolic paraboloid

$z(0,0) = 2(0)^2 + 3.(0)^2 = 0.$

the critical point is $(0,0,0)$.

Since the surface opens upwards and reaches its minimum value at $(0,0,0)$, it is a relative minimum.

$z(0,0) = 4 - 2.(0)^2 - 3.(0)^2 = 4$

the critical point is $(0,0,4)$.

Since the surface opens downwards and achieves its max. value at $(0,0,0)$, it is a relative maximum.

$z(0,0) = (0)^2 - 2.(0)^2 = 0.$

$(0,0,0)$: critical point.

$\Longrightarrow$

(see next page)

for hyperbolic paraboloid:

Since the surface curves upwards in the x-direction and downwards in the y-direction, $(0,0,0)$ is a saddle point. It does not represent a maximum or minimum because it is higher in one direction and lower in another. Also, f_x and f_y exist and

1) $f_x = 4x$, $f_y = 6y$ 2) $f_x = -4x$, $f_y = -6y$ 3) $f_x = 2x$, $f_y = -4y$

$$\Rightarrow \quad f_x(0,0) = f_y(0,0) = 0.$$

A critical point $P_0(x_0, y_0)$ is considered a saddle point of $f(x,y)$ if every open disk centered at P_0 contains points from the domain of f where $f(x,y) > f(x_0,y_0)$ and points where $f(x,y) < f(x_0,y_0)$.

The previous example emphasizes the need for a method to determine the nature of a critical point.

The second derivative test for one-variable functions states that if $f'(x) = 0$, a relative maximum occurs at $x = c$ if $f''(c) < 0$ and a relative minimum occurs if $f''(c) > 0$. If $f''(c) = 0$, the test is inconclusive. The concept also applies to functions of two variables.

Second Partials Test

Let $f(x,y)$ have a critical point at $P_0(x_0,y_0)$ with continuous second-order partial derivatives in a disk centered at (x_0,y_0). Let (det = determinant)

$$D = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = f_{xx}\cdot f_{yy} - (f_{xy})^2 .$$

Then, at P_0:

Relative maximum	Relative minimum	Saddle point
if $D(x_0,y_0) > 0$, $f_{xx}(x_0,y_0) < 0$ or $D(x_0,y_0) > 0$, $f_{yy}(x_0,y_0) < 0$	if $D(x_0,y_0) > 0$, $f_{xx}(x_0,y_0) > 0$ or $D(x_0,y_0) > 0$, $f_{yy}(x_0,y_0) > 0$	if $D(x_0,y_0) < 0$

If $D(x_0,y_0) = 0$, the test is inconclusive at P_0.

Ex. Find all relative extrema and saddle points of the function

$$g(x,y) = x^2 + 3xy + 3y^2 - 4x - 6y + 7 .$$

$$\left.\begin{aligned} g_x &= 2x + 3y - 4 \\ g_y &= 3x + 6y - 6 \end{aligned}\right\} \quad \begin{aligned} g_x = 0 &\Rightarrow 2x + 3y - 4 = 0 \quad \text{--- (1)} \\ g_y = 0 &\Rightarrow 3x + 6y - 6 = 0 \quad \text{--- (2)} \end{aligned}$$

From Eq. (2), we can simplify it $x + 2y - 2 = 0$

$\Rightarrow x = 2 - 2y$. Substitute it into Eq. (1):

$\Rightarrow 2(2-2y) + 3y - 4 = 0$

$\Rightarrow 4 - 4y + 3y - 4 = 0 \Rightarrow -y = 0 \Rightarrow y = 0.$

Substituting $y=0$ back into $x=2-2y$:

$\Rightarrow x = 2-2.(0) \Rightarrow x=2$.

So, the critical point is $(2,0)$.

$$\left.\begin{array}{l} g_{xx} = 2 \\ g_{xy} = 3 \\ g_{yy} = 6 \end{array}\right\} \Rightarrow D = g_{xx}\, g_{yy} - (g_{xy})^2 = 2 \cdot 6 - 3^2$$

$$\Rightarrow D = 12-9 = 3 .$$

Since $D=3>0$ and $g_{xx}=2>0$, then the critical point $(2,0)$ is a relative minimum.

Thus, the only relative extremum is a minimum at $(2,0)$.

Ex: Find all relative extrema and saddle points for the function $h(x,y) = x^3 - 3xy^2$.

$$\left.\begin{array}{l} h_x = 3x^2 - 3y^2 \\ h_y = -6xy \end{array}\right\} \Rightarrow \begin{array}{l} 3x^2 - 3y^2 = 0 \ \ldots (1) \\ -6xy = 0 \ \ldots (2) \end{array}$$

$-6xy = 0 \Rightarrow x=0$ or $y=0$.

- If $x=0$, substitute into Eq. (1) :
  $3.(0)^2 - 3y^2 = 0 \Rightarrow y=0$.
- If $y=0$, substitute into Eq. (1) :
  $3x^2 - 3.(0)^2 = 0 \Rightarrow x=0$.

The critical point is $(0,0)$.

$$\left.\begin{array}{l} h_{xx} = 6x \\ h_{yy} = -6x \\ h_{xy} = -6y \end{array}\right\} \Rightarrow \left.\begin{array}{l} h_{xx}(0,0) = 6.(0) = 0 \\ h_{yy}(0,0) = -6.(0) = 0 \\ h_{xy}(0,0) = -6.(0) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} D = h_{xx} \cdot h_{yy} - h_{xy}^2 \\ D = 0 \cdot 0 - 0^2 \\ D = 0 . \end{array}$$

Thus, $h(x,y)$ has a saddle point at $(0,0)$.

A function $f(x,y)$ reaches both an absolute maximum and an absolute minimum on any closed, bounded set S if it is continuous.

To determine the absolute extrema of a continuous function f on a closed, bounded set S, follow these steps:

1. Identify all critical points of f within S,
2. Locate all boundary points of S where absolute extrema may occur (including critical points and endpoints.),
3. Evaluate $f(x_0,y_0)$ for each point identified in steps 1 and 2.
4. The absolute maximum of f on S is the largest value from step 3, and the absolute minimum is the smallest value.

Ex. Find the absolute extrema of the function $f(x,y) = x^2 - y^2$ over the disk $x^2+y^2 \leq 1$.

1. $\left.\begin{array}{l} f_x = 2x \ , \quad f_x = 0 \\ f_y = -2y \ , \quad f_y = 0 \end{array}\right\} \Rightarrow$ critical point : $(0,0)$.

2. $f(0,0) = 0$ (evaluate at the critical point.)

3. $\left.\begin{array}{l} x = \cos\theta \\ y = \sin\theta \end{array}\right\} \Rightarrow f(\cos\theta, \sin\theta) = \cos^2\theta - \sin^2\theta = \cos(2\theta)$
(evaluate on the boundary)

4. $f_\theta = -2\sin(2\theta) \Rightarrow f_\theta = 0 \Rightarrow \sin(2\theta) = 0$
$\Rightarrow 2\theta = n\pi \Rightarrow \theta = \frac{n\pi}{2} \quad (n = 0,1,2,3)$
$\theta = 0 \Rightarrow f(1,0) = 1$, $\qquad \theta = \pi \Rightarrow f(-1,0) = 1$
$\theta = \frac{\pi}{2} \Rightarrow f(0,1) = -1$, $\qquad \theta = \frac{3\pi}{2} \Rightarrow f(0,-1) = -1$

Then, the absolute extrema of f over disk $x^2+y^2 \leq 1$ are

- absolute maximum : 1 at $(1,0)$ and $(-1,0)$,
- absolute minimum : -1 at $(0,1)$ and $(0,-1)$.

Ex. Find the absolute extrema of the function $f(x,y) = x^2+y^2-2xy$ over the disk $x^2+y^2 \leq 4$.

Hint: Use the method of critical points and evaluate on the boundary of the disk to determine the absolute extrema.

① $\left.\begin{array}{l} f_x = 2x-2y \\ f_y = 2y-2x \end{array}\right\} \Rightarrow \left.\begin{array}{l} f_x=0 \\ f_y=0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x=y \\ x=y \end{array}\right.$

Then, the critical points occur along the line $x=y$.

We check if there are any critical points inside the disk:

Substituting $y=x$ into $x^2+y^2 \leq 4 \Rightarrow x^2+x^2 \leq 4$

$\Rightarrow 2x^2 \leq 4 \Rightarrow x^2 \leq 2$. So, $x=y$ must lie within $x = \mp\sqrt{2}$. Thus, the critical point is $(x,y)=(\sqrt{2},\sqrt{2})$.

② $f(\sqrt{2},\sqrt{2}) = (\sqrt{2})^2 + (\sqrt{2})^2 - 2(\sqrt{2})\cdot(\sqrt{2}) = 2+2-4 = 0$.

③ $\left.\begin{array}{l} x=2\cos\theta \\ y=2\sin\theta \end{array}\right\} \Rightarrow f(2\cos\theta, 2\sin\theta) = (2\cos\theta)^2 + (2\sin\theta)^2 - 2(2\cos\theta)(2\sin\theta)$

$= 4\cos^2\theta + 4\sin^2\theta - 8\sin\theta\cos\theta = 4-4\sin(2\theta)$

④ $f'(\theta) = -8\cos(2\theta) \Rightarrow f'(\theta)=0 \Rightarrow \cos(2\theta)=0$

$\Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

$f(\frac{\pi}{4}) = 4-4\sin(\frac{\pi}{2}) = 4-4 = 0$

$f(\frac{3\pi}{4}) = 4-4\sin(\frac{3\pi}{2}) = 4-(-4) = 8$

$f(\frac{5\pi}{4}) = 4-4\sin(\frac{5\pi}{2}) = 4-4 = 0$

$f(\frac{7\pi}{4}) = 4-4\sin(\frac{7\pi}{2}) = 4-(-4) = 8$

- absolute max. of f is 8, occuring at $\frac{3\pi}{4}, \frac{7\pi}{4}$,
- absolute min. of f is 0, occuring at $(\sqrt{2},\sqrt{2})$ and $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$.

Ex. Find the point on the plane $2x-3y+z=8$ that is closest to the point $P(1,2,3)$.

If $Q(x,y,z)$ is a point on $2x-3y+z=8$, then we can write $z=8-2x+3y$ and the distance from P to Q is determined by

$$d=\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}.$$

Taking $f(x,y)=d^2$ (instead the minimizing d, we minimize the square of the distance), as the minimum of d occurs at the same points as the minimum of d^2. Substituting z into the distance formula:

$$f(x,y)=(x-1)^2+(y-2)^2+(8-2x+3y-3)^2$$
$$=(x-1)^2+(y-2)^2+(5-2x+3y)^2$$
$$=(x^2-2x+1)+(y^2-4y+1)+(25-20x+30y+4x^2-12xy+9y^2)$$

$$\Rightarrow f(x,y)=5x^2+10y^2-20x+30y-12xy+30.$$

1 $\left.\begin{array}{l} f_x=10x-20-12y \\ f_y=18y+30-12x \end{array}\right\} \Rightarrow$ 2 $\left.\begin{array}{l} f_x=0 \\ f_y=0 \end{array}\right\} \Rightarrow \begin{array}{l} 10x=12y+20 \Rightarrow x=\frac{12y+20}{10} \\ 18y=12x-30 \Rightarrow y=\frac{12x-30}{18} \end{array}$

3 $$y=\frac{2.\left(\frac{6y+10}{5}\right)-5}{3}=\frac{\frac{12y+20}{5}-5}{3}=\frac{12y-5}{15}$$

$$\Rightarrow y=\frac{12y-5}{15} \Rightarrow 15y=12y-5 \Rightarrow 3y=-5 \Rightarrow y=-\frac{5}{3}$$

$$\Rightarrow x=\frac{6y+10}{5}=\frac{6.\left(-\frac{5}{3}\right)+10}{5}=\frac{-10+10}{5}=0 \Rightarrow x=0$$

$\left(0,-\frac{5}{3}\right)$: critical point.

4 $z=8-2.(0)+3.\left(-\frac{5}{3}\right)=8-5=3$

5 $d=\sqrt{(1-0)^2+\left(2-\left(-\frac{5}{3}\right)\right)^2+(3-3)^2}=\sqrt{1+\frac{121}{9}}=\frac{\sqrt{130}}{3}.$ $\Big\}$ minimum distance

Ch. 11.8 Lagrange Multipliers

In applied problems, we often optimize a function of two variables under a constraint. For example, if $K(x,y,z)$ represents the concentration of a substance in a solution and $z=f(x,y)$ describes a reaction surface, we aim to find the maximum concentration on this surface, subject to the constraint $z=f(x,y)$.

Lagrange's theorem states that if f and g have continuous first partial derivatives and f has an extremum at $P(x_0,y_0)$ on the constraint curve $g(x,y)=c$, and $\nabla g(x_0,y_0) \neq 0$, then there exists a number λ such that $\boxed{\nabla f(x_0,y_0) = \lambda \cdot \nabla g(x_0,y_0)}$.

Method of Lagrange Multipliers :

Given that f and g satisfy Lagrange's theorem and $f(x,y)$ has an extremum subject to $g(x,y)=c$, follow these steps :

a) Solve the system of equations :

- $f_x(x,y) = \lambda \cdot g_x(x,y)$,
- $f_y(x,y) = \lambda \cdot g_y(x,y)$,
- $g(x,y) = c$.

b) Evaluate f at all solutions from (a) and on the boundary. The extremum will be among these values.

Ex. Given the function $f(x,y)=3x+4y$ and the constraint $x^2+y^2=16$, use the method of Lagrange multipliers to find the maximum and minimum value of f.

- $g(x,y)=x^2+y^2-16$ (define the constraint)
- $f_x(x,y)=3$ }
- $f_y(x,y)=4$ } (find the partial derivatives)

- $f_x=\lambda . g_x \Rightarrow 3=\lambda.(2x)$
- $f_y=\lambda . g_y \Rightarrow 4=\lambda(2y)$ } (from the system of eqs. using Lagrange's method)
- $x^2+y^2=16$ (the constraint)

$$\left.\begin{array}{l} 3=2\lambda x \Rightarrow \lambda=\frac{3}{2x} \\ 4=2\lambda y \Rightarrow \lambda=\frac{4}{2y}=\frac{2}{y} \end{array}\right\} \Rightarrow \frac{3}{2x}=\frac{2}{y}$$

$$\Rightarrow y=\frac{4}{3}x .$$

Substituting the constraint :

$$x^2+\left(\frac{4}{3}x\right)^2=16 \Rightarrow x^2+\frac{16}{9}x^2=16 \Rightarrow 25x^2=9.16$$

$$\Rightarrow x^2=\frac{144}{25} \Rightarrow x=\mp\frac{12}{5}$$

$$y=\frac{4}{3}x \Rightarrow \begin{cases} y=\frac{4}{3}\left(\frac{12}{5}\right)=\frac{16}{5}, & \left(\frac{12}{5},\frac{16}{5}\right) \\ y=\frac{4}{3}\left(-\frac{12}{5}\right)=-\frac{16}{5}, & \left(-\frac{12}{5},-\frac{16}{5}\right) \end{cases}$$

Evaluate f at the critical points :

$$f\left(\frac{12}{5},\frac{16}{5}\right)=3.\left(\frac{12}{5}\right)+4.\left(\frac{16}{5}\right)=\frac{36}{5}+\frac{64}{5}=\frac{100}{5}=20$$

$$f\left(-\frac{12}{5},-\frac{16}{5}\right)=3\left(-\frac{12}{5}\right)+4\left(-\frac{16}{5}\right)=-\frac{36}{5}-\frac{64}{5}=-\frac{100}{5}=-20$$

Some points on the circle $x^2+y^2=16$ are :

$(-4,0)$, $(4,0)$, $(0,4)$, and $(0,-4)$.

They are candidates for extreme values. Then ;

$f(4,0)=3.(4)+4.(0)=12$, $f(-4,0)=3(-4)+4.(0)=-12$,
$f(0,4)=3.(0)+4.(4)=16$, $f(0,-4)=3.(0)+4.(-4)=-16$.

Therefore, the maximum value of f is 20 at $\left(\frac{12}{5},\frac{16}{5}\right)$,
the minimum value of f is -20 at $\left(-\frac{12}{5},-\frac{16}{5}\right)$.